This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Phys 796.5

B

Harvard University

DEPARTMENT OF MATHEMATICS

FROM THE LIBRARY OF

GUSTAVUS HAY

(A.B. 1850, M.D. 1857)

OF BOSTON

GIFT OF MRS. HAY

MAY 16, 1908

Nov. 2, 1915
Transferred from
Mathematical Library

AN ELEMENTARY TREATISE ON MECHANICS.

TRANSLATED FROM THE FRENCH OF M. BOUCHARLAT.

WITH

ADDITIONS AND EMENDATIONS, DESIGNED TO ADAPT IT TO THE USE OF THE CADETS OF THE U. S. MILITARY ACADEMY.

BY EDWARD H. COURTENAY,

PROFESSOR OF NATURAL AND EXPERIMENTAL PHILOSOPHY IN THE ACADEMY.

NEW-YORK:
PRINTED AND PUBLISHED BY J. & J. HARPER,
NO 82 CLIFF-STREET,
AND SOLD BY THE BOOKSELLERS GENERALLY THROUGHOUT THE UNITED STATES.

1833.

Phys 796.5
B

Harvard University
Math. Dept. Library.
From the estate
of Dr. Gustavus Hay

[Entered according to the Act of Congress, in the year 1833, by J. & J. Harper, in the Office of the Clerk of the Southern District of New-York.]

PREFACE.

In preparing a translation of Boucharlat's Elements of Mechanics, it has been the principal object of the translator to supply a suitable text-book for the use of the Cadets of the United States' Military Academy. To accomplish this object more effectually, it has been deemed necessary to introduce several subjects which are not noticed in the original, and to extend or modify others, where the methods of investigation adopted by the Author appeared incomplete or obscure. It was also judged proper to omit one or two subjects, the discussion of which is usually reserved for works of a less elementary character.

These alterations were adopted with less hesitation as the work was principally designed for a special purpose; but it is believed that they will render the work more generally useful, by facilitating the comprehension of many of the more difficult investigations, and by affording much valuable information in relation to those subjects which were not discussed in the original, but which are generally admitted to form an essential part of an elementary course of Mechanics.

In supplying the deficiencies of the original, reference has been had most frequently to the works of Poisson, Francœur, Navier, Persy, Genieys, and Gregory; and in some few instances, the methods of investigation pursued by those authors have been adopted with but slight alterations.

The works of Boucharlat have long enjoyed an unusual share of public favour; and the hope is therefore entertained that the treatise now presented, in our own language, will prove a useful introduction to the study of the higher branches of Mechanics, and that it will be received with indulgence by all those who are disposed to cultivate a taste for the most interesting application of Mathematical Science.

As the entire work may be found to constitute too extensive a course for those students who can devote but a limited time to the study of Mechanics, it was thought expedient to indicate such of the more difficult subjects as might be omitted. These subjects are designated in the table of contents by being printed in italics; and they will be found to be unnecessary in enabling the student to comprehend those which follow.

CONTENTS.

PART FIRST.

STATICS.

Page
Introductory Remarks and Definitions 9
Of the Composition and Decomposition of Forces 11
Of Forces situated in the same Plane and applied to a single Point 20
General Remarks on Forces situated in any manner in Space 24
Of Forces situated in Space and applied to a Point 27
Of the conditions of Equilibrium of a Point acted upon by several Forces, and subjected to the condition of remaining upon a given Surface 31
Of the conditions of Equilibrium of a Point acted on by several Forces, and subjected to the condition of remaining constantly on two curved Surfaces, or on a Curve of double Curvature 36
Of Parallel Forces 39
Of Forces situated in the same Plane and applied to Points connected together in an invariable manner 48
Of Forces acting in any manner in Space 60
Theory of the Principal Plane, and Analogy existing between Projections and Moments 71
Centre of Gravity 79
Of the Centrobaryc Method 97
Machines—Cords 99
Of the Catenary 103
Of the Lever 109
Of the Pulley 116
Of the Wheel and Axle 120
Of the Inclined Plane 126
Of the Screw 128
Of the Wedge 131

Page
Friction 132
Effects of Friction in certain Machines 136
Of the Stiffness of Cordage 142
On the Resistance of Solids 145
Of the Resistance to Compression or Extension 147
Of the Resistance of a Solid to Flexure and Fracture produced by a Force acting at right angles to the direction of the Fibres 150
Of the Figure of the Solid after Flexure 164
Of Solids of equal Resistance 175
Of the Principle of Virtual Velocities 177
Of the Position of the Centre of Gravity of a System when in Equilibrio 184

PART SECOND.

DYNAMICS.

Of the Law of Inertia 187
Of uniform rectilinear Motion 188
Of varied Motion 190
Of uniformly varied Motion 194
Of the Motion of a Body projected vertically upward . . . 197
Of the vertical Motion of a Body when acted upon by the Force of Gravity considered as variable 199
Of the vertical Motion of a Body in a resisting Medium . . . 202
Of the Motions of Bodies upon Inclined Planes 205
Of curvilinear Motion 208
Of the Motion of a material Point when compelled to describe a particular Curve 221
Of the Motion of a material Point when compelled to move upon a curved Surface 229
Of the Motion of a material Point on the Arc of a Cycloid . . 236
Of oscillatory Motion 238
Of the Simple Pendulum 240
Of the Centrifugal Force 247
Of the System of the World 253
Of the Motions of Projectiles in Vacuo 272
Of the Motions of Projectiles in a resisting Medium 278
Of the different Methods of measuring the Effects of Forces 288

Page
Of the direct Impact of Bodies 293
Of the direct Impact of unelastic Bodies 293
Of the direct Impact of elastic Bodies 294
Of the Preservation of the Motion of the Centre of Gravity in the Impact of Bodies 297
Of the Preservation of living Forces in the Impact of elastic Bodies—Relative velocity before and after Impact—Loss of living Force in the Collision of unelastic Bodies . . . 299
Principle of D'Alembert 301
Of the Motion of a Body about a fixed Axis 309
Of the Moment of Inertia 316
Of the Motion of a Body about a fixed Axis when acted upon by incessant Forces 320
Of the Compound Pendulum 322
Of the Motions of a Body in Space when acted upon by impulsive Forces 328
Of the Motions of a System in Space when acted upon by incessant Forces 333
General Equations of the Motions of a System of Bodies . . 338
General Principle of the Preservation of the Motion of the Centre of Gravity 345

PART THIRD.

HYDROSTATICS.

Of the Pressure of Fluids 349
General Equations of the Equilibrium of Fluids 351
Application of the general Equations of Equilibrium to incompressible Fluids 354
Application of the general Equations of Equilibrium to elastic Fluids 359
Of the Pressure of heavy Fluids 361
Of the Equilibrium, Stability, and Oscillations of floating Bodies 366
Specific Gravity—Hydrostatic Balance—Hydrometer . . . 379
Of the Pressure and Elasticity of Atmospheric Air 384
Of Pumps for raising Water 386
Of the Air-pump 391
Of the Barometer 398

PART FOURTH.

HYDRODYNAMICS.

Page
Of the Discharge of Fluids through horizontal Orifices 407
Of the Motion of Water in Pipes 429

ELEMENTS OF MECHANICS.

PART FIRST.

STATICS.

INTRODUCTORY REMARKS AND DEFINITIONS.

1. **Mechanics** is the science which treats of the laws of equilibrium and motion. When applied to solid bodies it is divided into Statics and Dynamics; the former discussing the conditions of their equilibrium, and the latter those of their motion. In the application of Mechanics to the consideration of fluid substances a similar division is likewise made, viz. Hydrostatics, which treats of the equilibrium of fluids, and Hydrodynamics, which investigates the circumstances resulting from their motions.

2. The object proposed in Statics being the determination of the laws of equilibrium, this state of equilibrium may always be regarded as resulting from the mutual destruction of several forces.

3. The term *force* or *power* is applied to every cause which impresses on a body or a material point a motion or tendency to motion.

4. A force may act on a material point either by drawing the point towards it, or by pushing the point in advance of it. The first hypothesis will always be adopted, unless the contrary is expressly indicated.

5. A material point being solicited by a single force will naturally move in a right line, since there can be no reason why it should deviate to the right rather than to the left of this line.

6. The right line along which a force acts is called the line of direction.

7. The effect of a force depends, 1°. On its intensity; 2°. On its point of application; 3°. On its line of direction; and 4°. On its pushing or pulling along this line.

8. By the intensity of a force, we understand its greater or less capacity to produce motion.

9. If two forces directly opposed to each other sustain in equilibrio a material point or an inflexible right line, the intensity of either one of these forces may be assumed arbitrarily, provided we assign an equal intensity to the second force. A similar remark is equally applicable to a system composed of any number of forces; and hence it appears that the conditions of equilibrium will depend simply on the ratios of the forces, and not on their absolute intensities.

10. Having assumed one force as a unit of measure, we say that a second force is equal to it, when, if directly opposed to it, an equilibrium would ensue.

Two equal forces applied to a material point, acting along the same right line, and in the same direction, constitute a double force: in like manner a triple force may be regarded as resulting from the union of three equal forces, &c.; so that the number of these equal forces will constantly be proportional to their joint intensity. It may hence be inferred that if several forces solicit the point M (*Fig.* 1) in the same line of direction from M towards B, we can add into one sum all these forces, since their joint effect will be precisely the same as that of a single force equal to their sum. For the same reason we should subtract from this sum, or we should regard as negative all the forces which tend to solicit the point from M towards A.

11. The unit of force being arbitrary, it may be represented by any portion of its line of direction.

12. When a force is applied to any point of a body whose several parts are firmly connected together, this point cannot be put in motion without communicating the motion to the other parts of the body; if, therefore, a force be applied to any point A (*Fig.* 1), it will have the same effect as though it were applied to any other point M, assumed on the line of

direction AB. Moreover, if we drop the consideration of a body, we may still regard the points in space situated on the line of direction as mathematical points, no one of which can be moved without imparting its motion to all the others.

13. It appears from Art. 12, that by interposing a fixed obstacle on the line of direction of a force, its effect will be entirely overcome.

14. Two equal forces P and Q applied to the points A and B of an inflexible right line (*Figs.* 2 and 3), and acting along this line, but in contrary directions, will sustain each other in equilibrio: for if the force P tends to draw the point A from A towards a, the point B, which is firmly connected with A by means of the intermediate points, will have a tendency to describe the space Bb, equal to Aa; but by hypothesis, the force Q tends to move the point B over a space Bb' equal to Aa; and since B cannot yield to one of these influences rather than to the other, it must remain immoveable, and an equilibrium will necessarily ensue (Art. 13). In like manner, if the forces P and Q had been supposed to exert a tendency to push A and B, the same consequences might have been deduced.

15. When the right line AB is reduced to a point, the two equal forces, being directly opposed, are still in equilibrio; but if the forces are unequal, the point M (*Fig.* 1) will be moved in the direction of the greater, by a force equal to the difference of the two unequal forces.

Of the Composition and Decomposition of Forces applied to a Point.

16. When two forces act upon a moveable point in directions forming with each other an angle whose summit is the point of application, the state of equilibrium cannot subsist.

For, if we suppose the two forces P and Q (*Fig.* 4) to be in equilibrio, we may introduce a third force P′ equal and directly opposed to the force P. The forces P and Q being supposed to destroy each other, the force P′ must produce its entire effect, and must consequently move the point M in a direction from M towards P′. But P and P′, being equal and directly opposite, must likewise destroy each other, and the force Q will therefore act as though it were alone, soliciting the point M in a direction from M towards Q; and since it is impossible for the point M to move in two directions at the same time, we cannot suppose that P and Q are in equilibrio without involving an absurdity.

17. Since an equilibrium cannot subsist between two forces whose lines of direction are not coincident, the point M will tend to move in a certain direction MR, as though it were solicited by a single force R. This force is called the *resultant* of the two others, and the original forces are called *components.*

It may be observed that two forces which have a resultant do not always intersect. For example, if two parallel forces P and Q be supposed to act on a body, and if a third force R be found which shall produce the same effect, R will be the resultant of the forces P and Q.

18. Having examined the conditions of equilibrium of two equal forces acting on a point, the most simple case which next presents itself is that of three equal forces applied to the same point. Let P, Q, and R represent these forces; if they produce an equilibrium, their directions will divide the circumference of a circle whose centre coincides with the point of application, into three equal parts (*Fig.* 5): for since the same reasons may be adduced to prove that the point should tend to move in the direction of each of these forces, it follows that it cannot yield to the influence of either in preference, and must consequently remain at rest.

19. The equal angles PMQ, PMR, and QMR (*Fig.* 5), being measured by one-third of the entire circumference, each of them is equal to $\frac{4}{3}$ of a right angle, or 120°. Hence, if one of the three lines PM, QM, or RM be prolonged through M, it will bisect the angle formed by the other two. If MS, for example, be the prolongation of the line RM, the angles PMS, QMS will be equal, being supplements of the equal angles PMR and QMR; whence it appears that MS bisects the angle PMQ.

20. Let us next suppose the two equal forces P and Q (*Fig.* 6) to be applied perpendicularly to the extremities A

and B of a right line AB; the resultant of these forces will pass through the point O, the middle of the line AB, and will be equal in intensity to the sum of the intensities of the two forces P and Q. For, draw through the points A and B the four right lines AC, AD, BC, BD, each forming with AB an angle equal to $\frac{1}{3}$ of a right angle: the triangles ACB, ADB will be isosceles, and will have the sides AC, CB, AD, DB equal to each other.

The right lines AB, CD will intersect each other at right angles, and the figure ACBD will be a rhombus: the sides of this rhombus and their prolongations determine by their intersections the four obtuse angles ACB, ADB, P′AC, Q′BC, each of which is equal to $\frac{4}{3}$ of a right angle; for, the angle CAD being by construction equal to $\frac{2}{3}$ of a right angle, its supplement P′AC must be equal to $\frac{4}{3}$ of a right angle; and since the opposite sides of the rhombus are parallel, the angle ACB is equal to P′AC, and is consequently equal to $\frac{4}{3}$ of a right angle. The same may be proved of the angles CBQ′ and ADB. Moreover, since the line CD bisects the angle ACB, which was proved equal to $\frac{4}{3}$ of a right angle, it follows (Art. 19) that the three angles ACB, ACS, and BCS are equal to each other. In like manner it may be shown that there are three equal angles at each of the points A, B, and D.

21. We will now apply at the points A, B, C, D, which are supposed firmly connected together, twelve equal forces, distributed as follows:

At the point A three equal forces P, P′, P″,
At the point B three equal forces Q, Q′, Q″,
At the point C three equal forces S, S′, S″,
At the point D three equal forces V, V′, V″;

forming with each other angles equal to $\frac{4}{3}$ of a right angle: these twelve forces will sustain each other in equilibrio.

But the forces P′ and V″, Q′ and V′, being equal, and directly opposed, will destroy each other, as also will the forces P″ and S′, Q″ and S″. If, therefore, an equilibrium is maintained in the system, it must subsist between the four forces P, Q, S, and V. The two last, acting in the same direction along the line DC, are equivalent to a single force equal to their sum, which may be applied at O, a point in their line of direction. Thus, an equilibrium will take place between the forces P and Q and a force R whose line of direction passes through the middle of the line AB, and whose intensity is equal to the sum of the intensities of P and Q.

If we suppress P and Q, the equilibrium will be destroyed, but it may again be established by applying at O a single force R′ equal and directly opposed to the force R. The force R′ must therefore produce an effect precisely equal to the joint effect of P and Q, and will consequently be their resultant. We hence infer that *the resultant of two equal and parallel forces is equal to their sum, is parallel to them, and divides equally the line* AB, *which is drawn perpendicular to the common direction of those forces.*

22. To determine the resultant of two unequal parallel forces P and Q applied to the extremities A and B of a right line AB (*Fig.* 7), we will suppose p to represent the unit of force, and make $mp = \mathrm{P}$, $np = \mathrm{Q}$. The ratio of $m : n$ will be the same as that of the forces P and Q. Let the right line AB be also divided in the same ratio at the point D, and we shall have the proportion

$$\mathrm{P} : \mathrm{Q} :: \mathrm{AD} : \mathrm{DB} \ldots\ldots (a).$$

On the prolongations of AB, take $\mathrm{AA'} = \mathrm{AD}$, and $\mathrm{BB'} = \mathrm{BD}$; we shall then have, since A′D and DB′ are double AD and DB,

$$\mathrm{P} : \mathrm{Q} :: \mathrm{A'D} : \mathrm{DB'} :: m : n.$$

If then we divide A′D into m equal parts, DB′ will contain n such parts, and A′B′ will contain one of these parts as many times as p is contained in $\mathrm{P}+\mathrm{Q}$. And since any two of the points of division a', a'', a''', &c. separate three equal parts, while three points separate four parts, &c., the number of equal parts in the line A′B′ will exceed by unity the number of points of division. A force being applied at each point of division, there will remain one of the number $m+n$, of which one half may be applied at A′, and the other at B′; the several partial forces will thus be distributed throughout the line A′B′. But the points A′ and D being equally distant from the point A, the force $\frac{1}{2}p$ applied at A′ may be combined with one half of the force p applied at D, and their resultant, which is equal to their sum, will pass through A. The same remarks will apply to the forces p and p applied at a' and $a_{''}$, to the forces

p and p applied at a'' and $a_{''}$, &c.; thus, the total resultant of the partial forces distributed along A′D, will be equal to their sum P, and will pass through the point A. In like manner it may be shown that the forces applied to DB′ may be replaced by Q; and the entire system of partial forces may therefore be replaced by the two forces P and Q applied at the points A and B.

But these parallel forces may be otherwise compounded, by combining them in pairs taken at equal distances from the middle point O of the line A′B′; and it may thus be easily shown that the resultant of the whole system will pass through the point O, and will be equal to P+Q.

The position of the point O must now be determined. For this purpose, it may be remarked that A′O (*Fig.* 7), being one-half of A′B′, is equal to AB; and by substituting this value in the equation

$$AO = A'O - A'A,$$

which results immediately from an inspection of the figure, we shall obtain AO=AB−AA′, or AO=AB−AD=DB. In a similar manner it may be shown that OB=AD; and by substituting these values of DB and AD in the proportion (*a*), there will result

$$Q : P :: AO : OB \ldots\ldots (b).$$

If P and Q are incommensurable for the unit p, this proportion which results from the division of A′B′ into $m+n$ equal parts, might seem to fail: but by diminishing indefinitely the value of the unit p, and increasing in the same proportion the number of these divisions, the demonstration becomes applicable to all cases, since the equal parts Aa', $a'a''$, &c. being indefinitely small, the points of division will then become continuous.

23. This proposition is equally true when the two parallel forces P and Q are applied to the extremities of an oblique line CD (*Fig.* 8). For, by drawing AB at right angles to the common direction of the two forces, and transferring the points of application to the points A and B in their lines of direction, the proportion (*b*) will evidently subsist; but the similar triangles ACO, BDO, give AO : OB :: OC : OD; whence we obtain

Q : P :: OC : OD :

and we therefore infer that *when two parallel and unequal forces* P *and* Q *are applied to the extremities of a right line* CD, *their resultant will divide this line in the inverse ratio of the intensities of the forces.*

24. By the aid of this theorem we can readily demonstrate that of the parallelogram of forces, which may be enunciated as follows :—*If any two forces* P *and* Q *applied to a point* A (*Fig.* 9) *be represented in direction and intensity by the lines* AB *and* AC, *their resultant will be represented in direction and intensity by the diagonal of the parallelogram constructed upon the lines* AB *and* AC.

It is immediately obvious that the resultant will pass through the point of application of the forces; since the forces conspire to solicit this point, and their resultant, which may replace them, must therefore contain it.

25. The resultant of the two forces P and Q will likewise be contained in the plane of those forces. For, if it be situated above this plane, a position in all respects similar can be selected below the plane: the same arguments may then be advanced to prove that its direction coincides with either of these lines; and since the resultant cannot have two directions, we infer that it coincides with neither.

26. It may also be proved that the resultant of two equal forces (*Figs.* 10, 11, 12) will bisect the angle included between them.

For, if we suppose A*m* to represent the resultant of the two forces P and Q, and draw AD bisecting the angle PAQ, a line A*n* may always be found, whose position with respect to AD, AQ, and AP shall be precisely similar to that of A*m* with respect to AD, AP, and AQ; hence, the same reasons which would prove A*m* to be the resultant, become equally applicable to A*n*, and it might thence be inferred that there are two resultants: this being impossible, we conclude that the resultant coincides with AD.

27. Let the two unequal forces P and Q be now supposed to act upon the point A (*Fig.* 13), and let the parallelogram ABDC be constructed, whose sides AB and AC are taken on the lines of direction of those forces, and are proportional to their

intensities. It has already been shown that the resultant will pass through A, and it remains to be proved that it will also pass through D, the extremity of the diagonal AD. Having taken DE=AB=P,* draw EF parallel to AB, and apply at E and F, in contrary directions, the two forces Q′, Q″, each equal to Q. Since these forces will destroy each other, we can substitute for P and Q the four forces P, Q, Q′, and Q″. But by regarding P and Q′ as two parallel forces applied to the extremities of an inflexible line BE, and having obtained by construction the proportion

$$P : Q' :: DE : BD,$$

it follows immediately from the preceding theorem, that the resultant R of P and Q′ will pass through the point D. Again, if we transfer the force Q, and apply it at F, in its line of direction, the two equal forces Q and Q″ will have a resultant S, which, bisecting the angle QFQ″, will pass through D, the opposite angle of the rhombus CDEF. We thus obtain two forces R and S which are equivalent to the original forces P and Q; and since the forces R and S pass through the point D, the resultant of P and Q will likewise pass through the same point.

28. It will now be proved that if the intensities of the forces be represented by AB and AC, the diagonal AD will represent the intensity of the resultant (*Fig.* 14).

If at the point A (*Fig.* 14), and in the direction AD of the diagonal of the parallelogram constructed on the sides AB=P, AC=Q, there be applied a force X equal and directly opposed to the resultant of P and Q, an equilibrium will take place between the forces P, Q, and X. But we may regard Q as equal and directly opposed to the resultant of the forces P and X; hence it follows, that if through the extremity B of the line AB a line be drawn parallel to X, intersecting at

---

* It should be remarked that the expression AB=P is merely intended as an abridged method of stating that the line AB represents the relative intensity of the force P, when compared with the unit of force whose intensity is likewise represented by a line. In like manner, we speak of the "force AB," denoting thereby that the line AB represents the line of direction and relative intensity of the force. These abbreviations have been sanctioned by usage.

E the prolongation of the line AC, which, as has been already shown, coincides in direction with the diagonal of the parallelogram constructed on P and X, the line BE, being a side of this parallelogram, will be equal to the opposite side, which must represent X: but BE, being also the side of the parallelogram ED, is equal to the opposite side AD, which represents the diagonal of the parallelogram constructed upon P and Q; whence X=AD, and the intensity of the resultant is likewise measured by the length of the diagonal.

29. One of the simplest corollaries which may be deduced from the foregoing proposition is the trigonometrical relation existing between the components P and Q and their resultant R (*Fig.* 15). To obtain this relation, we will assume on the directions of these forces the parts AB and AC proportional to their intensities, and constructing the parallelogram ABDC, we shall have the proportion

$$P : Q : R :: AB : AC : AD.$$

And from the equality of the sides BD and AC, we shall have in the triangle ABD,

$$P : Q : R :: AB : BD : AD.$$

But the proportionality of the sides of the triangle to the sines of the opposite angles gives

$$AB : BD : AD :: \sin BDA : \sin BAD : \sin ABD.$$

Hence we deduce

$$P : Q : R :: \sin BDA : \sin BAD : \sin ABD.$$

The determination of the relations between P, Q, and R is thus reduced to the solution of a case in plane trigonometry.

30. If there be given, for example, the two components AB and AC, and the angle BAC contained between them, and it be required from these to determine the resultant, we shall have, in the triangle ABD, the sides AB, BD, and the angle B equal to the supplement of BAC. With these data we readily obtain the value of the side AD=R, by means of the formula

$$R^2 = P^2 + Q^2 - 2PQ \cos B.$$

If in this formula we wish to introduce the angle included between the two forces, since the angle B is the supplement of the angle BAD, we shall have the relation $\cos B = -\cos A$;

whence by substitution the following equation is obtained between the resultant, the two components, and the angle included between them,

$$R^2 = P^2 + Q^2 + 2PQ \cos A \ldots\ldots (1).$$

31. When the angle A becomes equal to 90°, the parallelogram ABDC (*Fig.* 16) becomes a rectangle, and $\cos A = 0$. The general relation between the resultant and its two components is then reduced to

$$R^2 = P^2 + Q^2.$$

The solution of the converse problem, or the resolution of a single force R into two components P and Q, having given directions, is readily effected by constructing a parallelogram upon the line representing the given force as a diagonal, the sides of the parallelogram having the directions of the required components.

32. When there are several forces lying in different planes, but all meeting in a single point, the resultant of the system can always be determined; for, by combining these forces in pairs, and substituting each resultant for its two components, the number of forces will be successively reduced, and we shall finally obtain but a single resultant.

33. The method of compounding any number of forces which has just been explained gives rise to a remarkable graphic construction. Thus, let P, P', P'', P''', &c. represent any forces whose directions intersect at the point A (*Fig.* 17), and whose intensities are expressed by the lines Ap, Ap', Ap'', Ap''', &c. assumed on the respective lines of direction; through the point p draw the line pr parallel and equal to the line Ap', and complete the parallelogram $Aprp'$; the diagonal $Ar = R$ will be the resultant of the forces P and P′: in like manner, by drawing rr' parallel and equal to Ap'', and forming the parallelogram $Arr'p''$, the diagonal Ar' will be the resultant of R and P″, and therefore the resultant of the three forces P, P′, and P″. By continuing this process, a polygon $Aprr'r''$ would be formed, having its sides parallel to the directions of the forces, and their lengths representing the intensities of those forces. The distances from the point A to the angles of this polygon will be

Ar=the resultant of P and P′,
Ar'=the resultant of P, P′, and P″,
Ar''=the resultant of P, P′, P″, and P‴.

And by repeating the construction for any number of forces, the distance from the point A to the extremity $r^{(n)}$ of the last side of the polygon will be equal to the resultant of the entire system.

Of Forces situated in the same Plane, and applied to a single Point.

34. Let P, P′, P″, &c. (*Fig.* 18) represent several forces situated in the same plane, their directions intersecting at the point A; through this point let there be drawn the rectangular axes Ax and Ay; then, denoting the respective intensities of these forces by AP, AP′, AP″, &c., let each be decomposed into two components, whose directions shall coincide with the rectangular axes.

For this purpose we will represent by α, α', α'', &c. the angles included between the forces and the axis of x, and by β, β', β'', &c. the angles which they form with the axis of y.

In the right-angled triangle ABC (*Fig.* 19), the side AC being expressed by AB cos A, and the side BC by AB sin A, the components of the forces P, P′, P″, &c. in the directions of the two axes are readily obtained: for the force P represented by AB, forming an angle α with the axis of x, and an angle β with the axis of y, will have for its components along these axes,

$$AC=P\cos\alpha,\ BC=P\cos\beta.$$

In like manner, the forces P′, P″, P‴, &c. will have for their components in the direction of Ax,

$$P'\cos\alpha',\ P''\cos\alpha'',\ P'''\cos\alpha''',\ \&c.,$$

and in the direction of the axis Ay,

$$P'\cos\beta',\ P''\cos\beta'',\ P'''\cos\beta''',\ \&c.$$

If the sum of the components acting in the direction of x be taken, as also the sum of those acting in the direction of y, we shall have, denoting these sums by X and Y respectively,

$$P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''+\&c.=X,$$
$$P\cos\beta+P'\cos\beta'+P''\cos\beta''+\&c.=Y;$$

and the entire system will thus be reduced (Art. 10) to two forces, of which one X is directed along the line Ax, the other Y acting along the line Ay. Calling R the resultant of these two forces, its value may be determined from the equation

$$X^2+Y^2=R^2.$$

35. For the purpose of rendering the preceding determination of the value of the resultant general, we have attributed the positive sign to all the cosines which enter into the expressions for X and Y; but it will be necessary in practice to regard the essential signs with which these quantities are severally affected. The following considerations will serve to explain the necessity of this distinction. Let a point M (*Fig.* 20) be solicited by a force represented in intensity by the line MP. By decomposing this force into two others whose directions shall coincide with the rectangular axes Mx and My, and calling α the angle which the direction of the force makes with the axis Mx, its two components will evidently become

$$MC=MP \sin \alpha,\ MD=MP \cos \alpha.$$

The forces which are directed in the line Mx, being regarded as positive when they act from M towards x, the component MD will obviously be positive. If the force MP should assume the position MP′, the angle α would be increased, and its cosine diminished; and if the angle becomes greater than 90°, the direction of the force will fall in the second quadrant. In this case it will assume the position MP″, and the cosine of the angle will change its sign. But it is evident that the component MD″ of the force MP″ becomes also negative, since it solicits the point M in a direction opposite to that in which it was urged by the component MD. Thus it appears that the signs of these two components result from the signs of the cosine of α, and hence the forces MP, MP′, &c., which solicit a point, may be always regarded as essentially positive, provided we attribute the appropriate signs to the cosines of the angles which they form with the axes.

36. If the force under consideration fall below AB, as in the position MP‴, the angle α being measured by the arc ALBP‴, will be greater than two right angles. To avoid this inconvenience, it has been agreed to reckon the angles α

and β indiscriminately on each side of their respective axes. Thus when the force falls beneath AB, the angle α will be measured, not by the arc ALBP‴, but by the arc AP‴, which has the same cosine. By this arrangement all the arcs employed are less than 180°. It is true that when the angle α is alone given, the direction of the force would appear indeterminate, since this angle may be counted either from A to P, or from A to P‴; but this ambiguity will immediately disappear by considering the value of the angle β, which is evidently acute for the force MP, but obtuse for the force MP‴.

37. Whatever may be the direction of the given force, since it must necessarily lie in one of the four right angles formed by the axes around the point M, its position must correspond to some one of those given in *Figs*. 21, 22, 23, 24.

In the first quadrant, α and β being acute give cos α positive, cos β positive,
In the second, ... α obtuse and β acute give cos α negative, cos β negative,
In the third, ... α obtuse and β obtuse give cos α negative, cos β negative,
In the fourth, ... α acute and β obtuse give cos α positive, cos β negative.

Each of these angles will be less than 180°.

38. It may be observed that the signs of these cosines are similar to those of the co-ordinates x and y of the point B. For example, if the point be situated within the angle $x'Ay$ (*Fig*. 22), x will be negative and y positive, while at the same time we shall have cos α negative and cos β positive.

39. For the purpose of making an application of the preceding principles, let us determine the resultant of the five forces P, P′, P″, P‴, P$^{\text{iv}}$, which are situated as represented in *Fig*. 25, and solicit the point A. By attributing to the components of the forces the positive or negative signs corresponding to the angles which are acute or obtuse, the components of

$$\left.\begin{matrix} P \\ P' \\ P'' \\ P''' \\ P^{\text{iv}} \end{matrix}\right\} \text{ will be } \left\{\begin{matrix} +P\cos\alpha, +P\cos\beta, \\ +P'\cos\alpha', -P'\cos\beta', \\ +P''\cos\alpha'', -P''\cos\beta'', \\ -P'''\cos\alpha''', -P'''\cos\beta''', \\ -P^{\text{iv}}\cos\alpha^{\text{iv}}, +P^{\text{iv}}\cos\beta^{\text{iv}}. \end{matrix}\right.$$

Having taken the sum of the components which act in one direction, we subtract from it the remaining components which act in an opposite direction, and we thus obtain

$$P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''-P'''\cos\alpha'''-P^{iv}\cos\alpha^{iv}=X,$$
$$P\cos\beta+P^{iv}\cos\beta^{iv}-P'\cos\beta'-P''\cos\beta''-P'''\cos\beta'''=Y.$$

40. If we defer the determination of the signs of the cosines until we wish to make an application of the preceding equations, the several terms may be written with the positive sign, and the general form of the equations will then become

$$P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''+\&c.=X\ .\ .\ .\ .\ .\ (2),$$
$$P\cos\beta+P'\cos\beta'+P''\cos\beta''+\&c.=Y\ .\ .\ .\ .\ .\ (3).$$

41. The resultant being represented by the diagonal of a rectangle, the lengths of whose sides are denoted by X and Y, its value will be determined by the equation

$$R=\sqrt{(X^2+Y^2)}\ .\ .\ .\ .\ .\ (4).$$

The position of the resultant remains to be determined. If we denote by a and b the angles which the resultant forms with the co-ordinate axes, we shall have

$$X=R\cos a,\ Y=R\cos b;$$

whence

$$\cos a=\frac{X}{R},\ \cos b=\frac{Y}{R}\ .\ .\ .\ .\ .\ (5).$$

The positions and intensities of the forces being given, the values of X and Y may be immediately deduced from the equations (2) and (3). These values being substituted in the equation (4), make known the value of the intensity of the resultant, and its position may be determined from the equations (5).

42. Its line of direction passing through the origin A (*Fig.* 26), will have for its equation

$$y=x\ \text{tang}\ a,\ \text{or}\ y=x\frac{\sin a}{\cos a};$$

and by substituting cos b for sin a, since a and b are complements of each other, the equation becomes

$$y=x\frac{\cos b}{\cos a},$$

and by substituting in this equation the values of cos a and cos b given in equations (5), we have

$$y=\frac{Y}{X}x.$$

43. When an equilibrium takes place, the intensity of the

resultant becomes equal to zero; and the formula (4) then assumes the form

$$\surd(X^2+Y^2)=0, \text{ or } X^2+Y^2=0.$$

But since every square is essentially positive, the preceding equation cannot be true, unless each of its terms is separately equal to zero; hence

$$X=0, \; Y=0.$$

Such are the equations which express the conditions of equilibrium of any number of forces situated in the same plane, and acting on a point.

44. If X alone were equal to zero, we should have

$$R=Y, \cos a=0, \cos b=\pm 1.$$

These equations prove that the resultant is equal to the component Y, and is directed along the axis of y.

In like manner it might be shown that if Y were equal to zero, the resultant would be equal to the component X, and would be directed along the axis of x.

General Remarks on Forces situated in any manner in Space.

45. If three forces solicit a point, their directions not being confined to a single plane, a theorem analogous to that of the parallelogram of forces will still serve to determine their resultant. Thus, let any three forces P, P′, and P″ be applied at the point A (*Fig.* 27), and let their intensities be represented by the lines AB, AC, and AD. If a parallelopiped be constructed upon these three lines, the diagonal AE, of the base of this parallelopiped, will evidently represent the resultant of the forces AB and AC; and by substituting the force AE for its two components, the resultant sought will be that of the forces AE and AD; it will therefore be terminated at the extremity F of the line EF drawn parallel and equal to the line AD; hence it will be the diagonal of the parallelopiped DE.

46. If the three forces are rectangular, the angle ABE will be a right angle, and hence we obtain

$$AE^2=AB^2+BE^2;$$

but the triangle AEF being also right-angled, we have

$$AF^2=AE^2+EF^2.$$

And by substituting for AE^2 its value given above, we deduce

$$AF^2=AB^2+BE^2+EF^2.$$

Or by replacing BE and EF by their equals AC and AD, we finally obtain

$$AF=\sqrt{(AB^2+AC^2+AD^2)},$$

or,

$$R=\sqrt{(P^2+P'^2+P''^2)},$$

the resultant of the three forces being denoted by R.

47. It has been shown that any number of forces lying in the same plane may always be referred to two rectangular axes: in like manner, we may refer to three rectangular axes those forces which are situated in different planes. Thus, having assumed three co-ordinate axes passing through any point O (*Fig.* 28), we draw through A, the point of application of a force P, the three rectangular axes Ax, Ay, and Az, parallel respectively to the axes of co-ordinates; and denoting by α, β, γ the angles formed by AD, the direction of the force P, with the three lines Ax, Ay, Az, the direction of the force will be determined when these angles become known.

48. The values of these angles may also be employed to determine the components of the force P, which act in directions parallel to the three co-ordinate axes. For, DC being perpendicular to the plane yAx, the angle DCA will be a right angle, and the triangle ADC, having the angle $D=\gamma$, will give

$$DC=AD\cos\gamma \;\ldots\ldots (6).$$

In like manner, the components parallel to Ax and Ay will be expressed by

$$AB=AD\cos\alpha,\ BC=AD\cos\beta \;\ldots\ldots (7).$$

And replacing the line AD by the force P which it represents, we obtain for the three rectangular components of P,

$$P\cos\alpha,\ P\cos\beta,\ P\cos\gamma.$$

49. It is important to observe that the values of two of the angles α, β, and γ will serve to determine that of the third. For, since the square of the diagonal AD is equal to the sum of the squares of the three edges, we have

$$AB^2+BC^2+DC^2=AD^2;$$

and substituting in this equation the values obtained from the equations (6) and (7), suppressing the common factor AD^2, there will remain

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1;$$

whence,

$$\cos\gamma=\pm\sqrt{(1-\cos^2\alpha-\cos^2\beta)}\ .....\ (8).$$

And since a similar value may be found for each of the other cosines, it follows that the angle formed by the direction of a force with either of the axes will become known, when the angles formed with the other two axes have been previously determined.

50. The radical in equation (8) being affected with the double sign, the cosine of γ may be either positive or negative. The first value will obtain when the angle is acute, and the second when it is obtuse.

But the angle γ will be acute or obtuse according to the position of the force P; in the first case, the force falls above the plane xAy, and the co-ordinates z of the points in the line representing the force, will therefore be positive; in the second, it falls below xAy, and the co-ordinates z will then be negative.

The same observations may be extended to the angles α and β considered with reference to the axes of x and y; so that in general the cosines will be affected with the same signs as the co-ordinates x, y, z, reckoned from A.

51. The signs of the cosines may also be determined by a rule which is founded on Art. 10. Thus, if Ax (*Fig.* 29) represent the line of direction of a component, this component will be positive when it acts in the direction from A towards x, but negative if it acts from A towards x'. The tendency of the force in the first case will be to remove the point A from the origin O, but in the second to cause its approach. Hence, we derive the following rule: *A component is positive when it tends to increase the co-ordinate of the point of application, and negative when it tends to diminish this co-ordinate.*

Of Forces situated in Space, and applied to a Point.

52. Let P, P′, P″, &c. represent different forces which solicit a point A, and let there be drawn through this point the three rectangular axes Ax, Ay, Az; represent by

α, β, γ, the angles formed by the force P with the axes of co-ordinates,
α', β', γ', the angles formed by P′ with the same axes,
α'', β'', γ'', the angles formed by P″ with the same axes,
&c. &c. &c.

By resolving these forces into components acting along the three axes, we shall obtain (Art. 48)

$P\cos\alpha$, $P\cos\beta$, $P\cos\gamma$, components of P,
$P'\cos\alpha'$, $P'\cos\beta'$, $P'\cos\gamma'$, components of P′,
$P''\cos\alpha''$, $P''\cos\beta''$, $P''\cos\gamma''$, components of P″.

If we defer, as in Art. 40, the determination of the signs of the cosines of these angles until the formulas are applied to a particular example, and denote by X, Y, and Z the components of the resultant, directed along the three axes, we shall have

$$P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''+\&c.=X\ldots\ldots(9),$$
$$P\cos\beta+P'\cos\beta'+P''\cos\beta''+\&c.=Y\ldots\ldots(10),$$
$$P\cos\gamma+P'\cos\gamma'+P''\cos\gamma''+\&c.=Z\ldots\ldots(11).$$

53. But X, Y, and Z being the projections AB, BC, and CD of the right line AD, which represents the resultant R (*Fig.* 28), we shall obtain (by Art. 46)

$$AB^2+BC^2+CD^2=AD^2,$$

and consequently,

$$X^2+Y^2+Z^2=R^2.$$

The intensity of the resultant will thus be determined, being expressed by the equation

$$R=\sqrt{(X^2+Y^2+Z^2)}\ldots\ldots(12).$$

Again, if we call a, b, and c the angles formed by the resultant with the co-ordinate axes, the components of R directed along the axes will be

$$R\cos a,\ R\cos b,\ R\cos c;$$

and since these components have been represented by the quantities X, Y, and Z, we shall have

$$X=R\cos a,\quad Y=R\cos b,\quad Z=R\cos c;$$

whence,

$$\cos a=\frac{X}{R},\quad \cos b=\frac{Y}{R},\quad \cos c=\frac{Z}{R}\ldots\ldots(13).$$

If the forces P, P′, P″, &c., and the angles α, β, γ, α', β', γ', &c. are known, the values of X, Y, and Z will result from the equations (9), (10), and (11). These values being substituted in formula (12), the intensity of the resultant will be determined, and its position will become known from the equations (13).

54. If an equilibrium subsists, the resultant becomes equal to zero, and the equation (12) then assumes the form

$$X^2+Y^2+Z^2=0.$$

And since this equation cannot be true unless the terms are separately equal to zero, we have

$$X=0,\ Y=0,\ Z=0.$$

These values reduce the equations (9), (10), (11) to

$$\left.\begin{array}{l} P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''+\&c.=0 \\ P\cos\beta+P'\cos\beta'+P''\cos\beta''+\&c.=0 \\ P\cos\gamma+P'\cos\gamma'+P''\cos\gamma''+\&c.=0 \end{array}\right\}\ldots\ldots(14).$$

Such are the conditions of equilibrium of a system of forces situated in any manner in space, and applied to a point.

55. If we determine the resultant of all the forces in the system except one, the remaining force will be found equal and directly opposed to this resultant. For, let R′ represent the resultant of all the forces except P; X′, Y′, and Z′ its three components, and a', b', and c' the angles which its direction forms with the co-ordinate axes; we shall have

$$\begin{aligned} X'&=P'\cos\alpha'+P''\cos\alpha''+P'''\cos\alpha'''+\&c., \\ Y'&=P'\cos\beta'+P''\cos\beta''+P'''\cos\beta'''+\&c., \\ Z'&=P'\cos\gamma'+P''\cos\gamma''+P'''\cos\gamma'''+\&c., \end{aligned}$$

and by means of these values the equations (14) may be reduced to

$$\begin{aligned} P\cos\alpha+X'&=0, \\ P\cos\beta+Y'&=0, \\ P\cos\gamma+Z'&=0; \end{aligned}$$

and eliminating X′, Y′, Z′, by the equations

$$X'=R'\cos a',\quad Y'=R'\cos b',\quad Z'=R'\cos c',$$

there results

$$\left.\begin{array}{l} P\cos\alpha=-R'\cos a' \\ P\cos\beta=-R'\cos b' \\ P\cos\gamma=-R'\cos c' \end{array}\right\} \ldots\ldots (15).$$

Taking the sum of the squares of these three equations, we obtain

$$P^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)=R'^2(\cos^2 a'+\cos^2 b'+\cos^2 c');$$

and since the second factor in each member is equal to unity, this equation reduces to

$$P^2=R'^2, \text{ or } P=R'.$$

The force P is regarded as essentially positive, its position being determined by the rule explained in Art. 35, &c

If the value of P be substituted in equations (15), the factor R′ being suppressed, those equations will become

$$\begin{array}{l} \cos\alpha=-\cos a' \ldots\ldots (16), \\ \cos\beta=-\cos b' \ldots\ldots (17), \\ \cos\gamma=-\cos c' \ldots\ldots (18). \end{array}$$

The relation between the values of cos α and cos a' indicates that a' and α are supplements of each other. For, if cos a' be represented by AC (*Fig.* 30), cos α will be represented by AC′=AC; whence a'=DAC, and α=D′AC.

But these two angles are supplements of each other; for, AC being equal to AC′, gives the angle DAC=D′AC′; whence, by substituting this value in the equation

$$D'AC+D'AC'=2 \text{ right angles},$$

we get

$$D'AC+DAC=2 \text{ right angles},$$

or the angles a' and α are supplements of each other.

In the same manner may it be proved by the equations (17) and (18), that the angles b' and β are supplements of each other, as also are the angles c' and γ.

It results from what precedes that the forces P and R′ are directly opposed; for, if R′ be supposed situated above the plane of x, y, having the co-ordinates x and y both positive, P will be situated below this plane, and will have the co-ordinates x and y both negative.

56. After reducing all the forces to three rectangular components X, Y, Z, it was shown that the resultant R would be represented by the diagonal of a parallelopiped, whose contiguous edges were respectively equal to X, Y, and Z (*Fig.* 27). The equation of this resultant, which is represented by AF, will therefore be that of a right line passing through A, the origin of co-ordinates, and through the point F, whose co-ordinates are equal to X, Y, and Z.

57. The case may be rendered yet more general by supposing that the point of application of the forces has the three co-ordinates x', y', and z'; the co-ordinates of the point F will then become (*Fig.* 31)

$$x'+X,\ y'+Y,\ z'+Z.$$

And the equations of the resultant, being that of a right line in space, will be of the form

$$z=ax+b,\quad z=a'y+b'\ \ldots\ldots\ (19):$$

substituting in these equations the co-ordinates of the point F, in place of the quantities x, y, and z, we find

$$z'+Z=ax'+aX+b,\quad z'+Z=a'y'+a'Y+b'\ \ldots\ldots\ (20);$$

but the co-ordinates of the point A should also satisfy the equations (19), and therefore we obtain

$$z'=ax'+b,\quad z'=a'y'+b'\ \ldots\ldots\ (21).$$

Subtracting these last from equations (20), we have

$$Z=aX,\quad Z=a'Y;$$

whence,

$$a=\frac{Z}{X},\quad a'=\frac{Z}{Y}.$$

Again, by eliminating b and b' between the equations (19) and (21), we find

$$z-z'=a(x-x'),\quad z-z'=a'(y-y'):$$

and by substituting the values of a and a' previously obtained, the equations of the resultant finally become

$$z-z'=\frac{Z}{X}(x-x'),\quad z-z'=\frac{Z}{Y}(y-y').$$

Of the Conditions of Equilibrium of a Point acted upon by several Forces, and subjected to the Condition of remaining upon a Given Surface.

58. The material point to which the forces P, P′, P″, &c. were applied, has been supposed hitherto to submit freely to the action which those forces exert; but if, on the contrary, the point were required to remain constantly on a given surface, the equations (14) would no longer be applicable, and the condition of the resultant being equal to zero, which was then necessary, would, under this supposition, be replaced by the condition that the resultant must be normal to the given surface. For, if the direction of the resultant be oblique to the surface, it can be decomposed into two forces, of which one shall coincide with the direction of the tangent, and the other with the normal: the first would cause the material point to slide along the surface, while the second would be overcome by the reaction of the surface. Hence, it follows that the resultant of all the forces must act on the point in the direction of the normal to the surface, and since the resultant is destroyed by the resistance of the surface, we may regard this resistance as a force directly opposed to the normal force, and denote its intensity by a quantity N.

If the intensity of the force N and the angles θ, θ', θ'', which it forms with the co-ordinate axes, were known, it would be sufficient to add to the equations of equilibrium the components N cos θ, N cos θ', N cos θ'' of the force N; we should thus obtain the equations of equilibrium

$$\begin{aligned}
&\mathrm{N}\cos\theta+\mathrm{P}\cos\alpha+\mathrm{P}'\cos\alpha'+\mathrm{P}''\cos\alpha''+\&\text{c.}=0,\\
&\mathrm{N}\cos\theta'+\mathrm{P}\cos\beta+\mathrm{P}'\cos\beta'+\mathrm{P}''\cos\beta''+\&\text{c.}=0,\\
&\mathrm{N}\cos\theta''+\mathrm{P}\cos\gamma+\mathrm{P}'\cos\gamma'+\mathrm{P}''\cos\gamma''+\&\text{c.}=0.
\end{aligned}$$

59. These equations may be simplified by representing, as in Art. 52, by X, Y, and Z, the sums of the components parallel to the three axes; the equations will thus become

$$\mathrm{N}\cos\theta+\mathrm{X}=0,\quad \mathrm{N}\cos\theta'+\mathrm{Y}=0,\quad \mathrm{N}\cos\theta''+\mathrm{Z}=0\ \ldots\ldots (22).$$

60. To determine the values of the unknown quantities cos θ, cos θ'; cos θ'', and N, we will suppose $\mathrm{L}=0$ to be the equation of the given surface, and x', y', and z' the co-ordinates of the

material point to which the forces are applied, and which by hypothesis is required to remain on this surface. The normal being a right line passing through the point whose co-ordinates are x', y', z', its equations will be of the form

$$x-x'=a(z-z'),\ y-y'=b(z-z') \ldots\ldots (23).$$

The differences $x-x'$, $y-y'$, $z-z'$, which enter into these equations, represent the projections of the right line on the axes of co-ordinates. To determine the relations existing between these projections and the angles θ, θ', θ'', let MN (*Fig.* 32) represent the right line in space referred to the co-ordinate axes whose origin is at the point O, and denote by x, y, z, x', y', z', the co-ordinates of the points N and M: if a plane DF be passed through the co-ordinates $\mathrm{MD}=z'$ and $\mathrm{BD}=y'$, and a second plane EG through $\mathrm{NE}=z$ and $\mathrm{EC}=y$, these two planes will be parallel to that of y, z, and the distance between them will be measured by the part $\mathrm{BC}=x-x'$ intercepted on the axis of x: but since every parallel to this axis is likewise perpendicular to the two planes, it follows that by drawing through the point M, the extremity of the co-ordinate z', the parallel MP to the axis of x, this parallel will be perpendicular to the plane EG, and will intersect it at a distance $\mathrm{MP}=x-x'$.

But, by connecting the point P with N, the point at which the right line MN intersects the plane EG, a triangle will be formed right-angled at P, since MP is perpendicular to the plane EG. Hence,

$$\mathrm{MP}=\mathrm{MN}\cos \mathrm{M},$$

or,

$$x-x'=\mathrm{MN}\cos\theta;$$

but MN being a right line passing through the two points whose co-ordinates are x, y, z, x', y', z', its length will be expressed by

$$\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}.$$

Substituting this value in the preceding equation, we deduce

$$\cos\theta=\frac{x-x'}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}}.$$

In like manner, by drawing planes through the co-ordinates x', z', and x, z, parallel to the plane of x, z, and through x', y', and x, y, parallel to the plane of x, y, we shall find for $\cos\theta'$ and $\cos\theta''$, the similar expressions

$$\cos\theta' = \frac{y-y'}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}},$$
$$\cos\theta'' = \frac{z-z'}{\sqrt{[(x-x')^2+(y-y')^2+(z-z')^2]}}:$$

by eliminating the values of $x-x'$, $y-y'$, by means of equations (23), and suppressing the common factor $z-z'$, we obtain

$$\left.\begin{array}{l} \cos\theta = \dfrac{a}{\sqrt{(a^2+b^2+1)}},\ \cos\theta' = \dfrac{b}{\sqrt{(a^2+b^2+1)}}, \\ \cos\theta'' = \dfrac{1}{\sqrt{(a^2+b^2+1)}}. \end{array}\right\} \ldots\ldots(24).$$

61. These values, which serve to determine the direction of the normal, contain the quantities a and b, which are yet unknown. The values of these quantities will now be determined. Let $L=0$ be the equation of the surface which passes through the point x', y', z'; if we draw through this point a plane tangent to the surface, the equation of this plane will be of the form

$$Ax+By+Cz+D=0;$$

and since it must be satisfied by the co-ordinates x', y', z', we shall have

$$Ax'+By'+Cz'+D=0.$$

Eliminating D between these two equations, the equation of the tangent plane to the surface becomes

$$A(x-x')+B(y-y')+C(z-z')=0;$$

and dividing by C, it may be put under the form

$$\frac{A}{C}(x-x')+\frac{B}{C}(y-y')+(z-z')=0 \ldots\ldots (25).$$

But if the plane be tangent to the surface whose equation is $L=0$, the values of $\frac{dz'}{dx'}$ and $\frac{dz'}{dy'}$ deduced from that equation, will be expressed as follows:

$$\frac{dz'}{dx'}=-\frac{A}{C},\ \frac{dz'}{dy'}=-\frac{B}{C} \ldots\ldots (26).$$

And from the known principles of analytical geometry, when a plane whose equation is $Ax+By+Cz+D=0$ is perpendicular to a right line represented by the equations $x=az+\alpha$, $y=bz+\beta$, the following relations between the constants exist:

$$\frac{A}{C}=a,\quad \frac{B}{C}=b:$$

the equations (26) will therefore reduce to

$$\frac{dz'}{dx'}=-a,\quad \frac{dz'}{dy'}=-b\ \ldots\ldots\ (27).$$

62. The values of these coefficients must now be determined from the equation of the surface. We obtain by differentiating,

$$\frac{dL}{dx}dx+\frac{dL}{dy}dy+\frac{dL}{dz}dz=0\,;$$

whence we infer that

$$dz=-\frac{\frac{dL}{dx}}{\frac{dL}{dz}}dx-\frac{\frac{dL}{dy}}{\frac{dL}{dz}}dy\,;$$

and by applying this equation to the point of tangency, for which the co-ordinates are x', y', z', we find

$$\frac{dz'}{dx'}=-\frac{\frac{dL}{dx'}}{\frac{dL}{dz'}},\quad \frac{dz'}{dy'}=-\frac{\frac{dL}{dy'}}{\frac{dL}{dz'}}\,;$$

substituting these values in the equations (27), they become

$$a=\frac{\frac{dL}{dx'}}{\frac{dL}{dz'}},\quad b=\frac{\frac{dL}{dy'}}{\frac{dL}{dz'}}.$$

Replacing a and b in equations (24), by their values found above, we obtain, after reduction,

$$\cos\theta=\pm\frac{\frac{dL}{dx'}}{\sqrt{\left\{\left(\frac{dL}{dx'}\right)^2+\left(\frac{dL}{dy'}\right)^2+\left(\frac{dL}{dz'}\right)^2\right\}}},$$

$$\cos\theta'=\pm\frac{\frac{dL}{dy'}}{\sqrt{\left\{\left(\frac{dL}{dx'}\right)^2+\left(\frac{dL}{dy'}\right)^2+\left(\frac{dL}{dz'}\right)^2\right\}}},$$

$$\cos\theta''=\pm\frac{\frac{dL}{dz'}}{\sqrt{\left\{\left(\frac{dL}{dx'}\right)^2+\left(\frac{dL}{dy'}\right)^2+\left(\frac{dL}{dz'}\right)^2\right\}}}.$$

The double sign is here prefixed to the values of $\cos\theta$, $\cos\theta'$, $\cos\theta''$, for the purpose of indicating that the resistance opposed by the surface may be exerted either in the direction of the normal or along its prolongation, according as the body is placed on the concave or convex side of the surface. The form of these equations being inconvenient for the purposes of calculation, they may be simplified by making

$$\pm\frac{1}{\sqrt{\left\{\left(\frac{dL}{dx'}\right)^2+\left(\frac{dL}{dy'}\right)^2+\left(\frac{dL}{dz'}\right)^2\right\}}}=V\ldots\ldots(28);$$

which reduces them to

$$\cos\theta=V\frac{dL}{dx'},\quad \cos\theta'=V\frac{dL}{dy'},\quad \cos\theta''=V\frac{dL}{dz'};$$

substituting these values of the cosines in equations (22), we obtain

$$NV\frac{dL}{dx'}+X=0,\ NV\frac{dL}{dy'}+Y=0,\ NV\frac{dL}{dz'}+Z=0\ldots\ldots(29).$$

63. The value of N remains to be determined. If we transpose X, Y, and Z in the equations (29), and take the sum of the squares of the three equations, we shall obtain

$$N^2V^2\left\{\left(\frac{dL}{dx'}\right)^2+\left(\frac{dL}{dy'}\right)^2+\left(\frac{dL}{dz'}\right)^2\right\}=X^2+Y^2+Z^2,$$

and reducing by means of equation (28) there results

$$N^2=X^2+Y^2+Z^2,$$

whence,

$$N=\sqrt{(X^2+Y^2+Z^2)}\ldots\ldots(30).$$

This value of N is precisely the same as that of the resultant of the entire system; but its components should be affected with signs contrary to those of the components of the resultant, since its action is exerted in an opposite direction. Thus, having determined the resultant of all the forces P, P', P'', &c., the reaction of the surface will be equal to this resultant, but will be exerted in an opposite direction.

64. If the direction of the normal force be parallel to the axis of z, we shall have

$$\theta=90^\circ,\ \theta'=90^\circ,\ \theta''=0, \text{ or } \theta''=180^\circ;$$

whence

$$\cos\theta=0,\ \cos\theta'=0,\ \cos\theta''=\pm 1:$$

and the equátions (22) will therefore reduce to

$$X=0,\ Y=0,\ N\pm Z=0;$$

which prove that the components in the direction of the tangent plane destroy each other, and that the reaction of the surface in the direction of the normal is equal to the sum of the components directed along the axis of z.

65. The nature of the problem may also be such that having given the forces P, P′, P″, &c. and the equation of the surface upon which the material point should rest, it might be required to determine x', y', and z', the co-ordinates of the point at which the forces should be applied in order that the material point should be sustained in equilibrio.

To resolve this problem, we first eliminate the quantity N, by combining the equations (29); the factor V will likewise disappear, and we shall then have

$$Z\frac{dL}{dx'}=X\frac{dL}{dz'},\ Z\frac{dL}{dy'}=Y\frac{dL}{dz'};$$

these equations, in conjunction with that of the surface, will serve to determine the co-ordinates x', y', and z' of the point of application.

Of the Conditions of Equilibrium of a Point acted on by several Forces, and subjected to the Condition of remaining constantly on two Curved Surfaces, or on a Curve of Double Curvature.

66. If a material point be retained on two curved surfaces, it cannot remain in equilibrio unless the force which solicits it can be decomposed into two components which shall be respectively normal to the given surfaces; for, if one of these components had a different direction, it might be decomposed into two forces, of which the first normal to one of the surfaces, should be destroyed by the reaction of the surface, and

the second tangent to the same surface, would move the body along the surface.

Let N and M represent the reactions of the two surfaces, and θ, θ', θ'', η, η', η'' the angles formed by their normals with three rectangular axes drawn through the point to which the forces are applied: by adopting the same course of reasoning as in Art. 59, we shall obtain

$$\left.\begin{array}{l} \mathrm{N}\cos\theta + \mathrm{M}\cos\eta + \mathrm{X} = 0 \\ \mathrm{N}\cos\theta' + \mathrm{M}\cos\eta' + \mathrm{Y} = 0 \\ \mathrm{N}\cos\theta'' + \mathrm{M}\cos\eta'' + \mathrm{Z} = 0 \end{array}\right\} \ldots\ldots (31).$$

The equations of the surfaces L=0 and K=0 being differentiated, make known, as in Art. 62, the values of the quantities $\cos\theta$, $\cos\theta'$, $\cos\theta''$, $\cos\eta$, $\cos\eta'$, $\cos\eta''$, and by adopting abbreviations similar to those of Art. 62, making

$$\pm\frac{1}{\sqrt{\left\{\left(\frac{d\mathrm{L}}{dx'}\right)^2+\left(\frac{d\mathrm{L}}{dy'}\right)^2+\left(\frac{d\mathrm{L}}{dz'}\right)^2\right\}}}=\mathrm{V},$$

and

$$\pm\frac{1}{\sqrt{\left\{\left(\frac{d\mathrm{K}}{dx'}\right)^2+\left(\frac{d\mathrm{K}}{dy'}\right)^2+\left(\frac{d\mathrm{K}}{dz'}\right)^2\right\}}}=\mathrm{U},$$

we shall find

$$\cos\theta=\mathrm{V}\frac{d\mathrm{L}}{dx'},\quad \cos\eta=\mathrm{U}\frac{d\mathrm{K}}{dx'},$$

$$\cos\theta'=\mathrm{V}\frac{d\mathrm{L}}{dy'},\quad \cos\eta'=\mathrm{U}\frac{d\mathrm{K}}{dy'},$$

$$\cos\theta''=\mathrm{V}\frac{d\mathrm{L}}{dz'},\quad \cos\eta''=\mathrm{U}\frac{d\mathrm{K}}{dz'}:$$

which values, being substituted in the equations(31), give

$$\left.\begin{array}{l} \mathrm{NV}\frac{d\mathrm{L}}{dx'}+\mathrm{MU}\frac{d\mathrm{K}}{dx'}+\mathrm{X}=0 \\ \mathrm{NV}\frac{d\mathrm{L}}{dy'}+\mathrm{MU}\frac{d\mathrm{K}}{dy'}+\mathrm{Y}=0 \\ \mathrm{NV}\frac{d\mathrm{L}}{dz'}+\mathrm{MU}\frac{d\mathrm{K}}{dz'}+\mathrm{Z}=0 \end{array}\right\} \ldots\ldots (32).$$

From these three equations the unknown quantities M and N may be readily eliminated; and since U and V enter into them in the same manner as M and N, they will also disappear in the elimination: or, to simplify the case, we may regard MU and NV as the unknown quantities, which, being eliminated between the three preceding equations, will give an equation of condition including one or more of the three variables. This resulting equation being combined with those of the surfaces, viz. $L=0$, $K=0$, will determine the co-ordinates x', y', z', of the point sought.

It may be proper to remark that the radicals, which would serve to complicate the expressions, disappear at the same time as the quantities U and V.

67. When the point is subjected to the condition of remaining on a curve of double curvature, such curve may be regarded as being formed by the intersection of two curved surfaces. The equations of these surfaces being represented as above by $L=0$ and $K=0$, the co-ordinates of the points in which they intersect will necessarily appertain to both surfaces, and the quantities x', y', and z' may therefore be regarded as having the same values in each of these equations; but we have also the equation of condition referred to in Art. 66; thus by eliminating the values of two of the co-ordinates, the third will be expressed in functions of known quantities: denoting by A, B, and C the values of the functions corresponding to each of the co-ordinates x', y', and z', we shall have

$$x'=A, \quad y'=B, \quad z'=C.$$

68. It may occur that the equation resulting from the elimination of M and N will not contain either of the variables. This case presents itself when the surfaces become planes; their equations $L=0$ and $K=0$ may then be put under the form $Ax+By+Cz+D=0$, and the differential coefficients are then constant. Under such circumstances the values of the intensities M and N determined by the equations (32) become independent of the co-ordinates x', y', and z'; and since these co-ordinates still apply to any points common to the two planes, it follows that the conditions of equilibrium will be fulfilled, if the forces be applied to any point whatever in the common intersection of the two planes. A similar remark is applicable to Art. 65.

Of Parallel Forces.

69. The forces which have been considered in the preceding paragraphs were supposed to have a common point of application; but if they were applied to different points of a body or system of bodies, the points being retained at fixed distances by means of their connexion with the intermediate points, we might regard the forces as having their points of application united by means of inflexible right lines.

70. Let there be two parallel forces P and Q applied to the extremities of a right line AB (*Fig.* 34), which intersects at right angles their common direction. It has been proved (Art. 22) that the intensity of the resultant of these forces will be equal to the sum of the intensities of the two components, and that its point of application O will divide the line AB in the inverse ratio of the two forces. This proposition may be demonstrated in another manner, provided we admit that of the parallelogram of forces, which is susceptible of direct proof.

Let the two parallel forces be represented by the right lines AP and BQ proportional to their intensities (*Fig.* 33); we can add to the system, without changing the value of the resultant, the two equal and opposite forces AM and BN, and the four forces AP, AM, BQ, and BN may then be replaced by the two AD and BI, the diagonals of the rectangles MP and NQ. But since these diagonals intersect at the point C, the forces AD and BI may be conceived to be applied at that point, and will be represented by CE=AD and CF=BI. If the forces CE and CF be then decomposed into rectangular components, by constructing the rectangles GL and HK, having their sides respectively equal and parallel to those of the rectangles MP and NQ, we shall replace CE and CF by the four forces CL, CK, CG, and CH. But the last two are equal, being equivalent to the forces AM and BN, which by hypothesis are equal, and being directly opposed, they must mutually destroy each other; there will therefore remain at the point C, the two forces CL and CK equal respectively to P and Q, and having the common direction of the line CO. The resultant of these two forces must evidently be equal to their sum; and if it be denoted by R, we shall have the relation

$$R=P+Q:$$

but since the resultant may be applied at any point in its line of direction, we will consider it as acting at O, the point in which it intersects the line AB; the position of this point may be determined thus: the two similar triangles CAO, CEL give the proportion

$$CO : AO :: CL : EL,$$

and the triangles COB, CKF give

$$BO : CO :: KF : CK;$$

whence, by multiplication, suppressing the common factor CO, we have

$$BO : AO :: CL \times KF : EL \times CK.$$

But KF and EL, being equal to BN and AM, which by hypothesis are equal to each other, the proportion reduces to

$$BO : AO :: CL : CK:$$

and since CL and CK are equivalent to the lines AP and BQ, which represent the intensities of the given forces, the proportion may be written

$$BO : AO :: P : Q \ldots\ldots (33).$$

Hence we conclude that the point of application O of the two parallel forces P and Q divides the line AB into two parts, reciprocally proportional to the intensities of those forces.

71. From the above proportion we obtain (*Fig.* 34)

$$BO+AO : AO :: P+Q : Q,$$

or,

$$AB : AO :: R : Q \ldots\ldots (34).$$

And from the equations (33) and (34) combined, we find

$$P : Q : R :: BO : AO : AB;$$

from which we derive the following rule: *The parts* AO, BO, *and* AB *comprised between any two of the forces* P, Q, *and* R, *will be constantly proportional to the third force.* The term R, for example, in the above proportion, corresponds to the portion AB, which is included between the forces P and Q.

72. If from the known values of P, Q, and AO, it were required to determine that of BO, the proportion would give

$$Q : P :: AO : BO;$$

whence,

$$BO = \frac{P \times AO}{Q}.$$

73. Reciprocally, if there were given the force R applied at O, and we wished to resolve it into two parallel components whose points of application should be A and B; by denoting the unknown components by P and Q, the value of the first would result from the proportion

$$AB : BO :: R : P;$$

and that of the second would in like manner be obtained by means of the proportion

$$AB : AO :: R : Q.$$

From these two proportions we deduce

$$P = \frac{R \times BO}{AB}, \quad Q = \frac{R \times AO}{AB}.$$

In the preceding demonstration, the forces P and Q have been supposed perpendicular to the line AB; but if they were oblique to the direction of this line, we might draw through O, the point of application of the resultant (*Fig.* 35), the right line CD, perpendicular to the direction of the given forces, and the force P applied at A would have the same effect as though it were applied at the point C. In like manner, the point of application of the force Q may be transferred from B to D: and since we have the proportion

$$P : Q :: OD : OC,$$

we shall likewise obtain from the similarity of the triangles OBD, AOC,

$$P : Q :: BO : AO.$$

74. When the forces P and Q act in opposite directions, the resultant is equal to the difference of these forces. For, let S (*Fig.* 36) be the resultant of the forces P and R, which are supposed to act in the same direction, we shall then have

$$S=P+R\ldots\ldots(35);$$

and if we replace S by a force Q equal in intensity, and directly opposed to it, an equilibrium will subsist between the three forces P, R, and Q: we may therefore regard R as being equal and directly opposite to the resultant of the forces P and Q, and the equation (35) will give for the intensity of this resultant

$$R=S-P;$$

but S and Q being equal in intensity, we have, by substituting the value of S,

$$R=Q-P.$$

The point O at which the resultant is applied, may be found by the proportion

$$AB:BO::R:Q,$$

whence we obtain

$$BO=\frac{AB\times Q}{R};$$

or, replacing R by its equal Q—P, we have

$$BO=\frac{Q\times AB}{Q-P}.$$

From this value of the distance BO, we infer that the point O will be farther removed from B in proportion to the diminution of the quantity Q—P; if therefore Q and P become equal, BO becomes infinite, and R becomes equal to zero: hence, if two parallel and equal forces act in contrary directions, but are not directly opposed, the equilibrium cannot be established except by the application of an infinitely small force at a point whose distance is infinite; it is therefore impossible in such cases to find a single finite force which shall sustain in equilibrio the two forces P and Q; or, in other words, the two forces P and Q cannot be replaced by a single resultant. The effect of these forces will be simply to turn the line AB about its middle point C.

75. These pairs of parallel and equal forces, acting in contrary directions, but not directly opposed, are called *couples.*

76. The results obtained in the preceding articles may be applied to any number of forces. Thus, let P, P', P", P‴, P"", (*Fig.* 37) represent parallel forces applied to the points A, B,

C, D, E, which are connected together by inflexible right lines; the point of application and the intensity of the resultant may be readily found. For, the forces P and P′ being compounded, their resultant will be applied at a point M, whose position may be determined by the following proportion,

$$AB : AM :: P+P' : P';$$

whence,

$$AM=\frac{AB\times P'}{P+P'}:$$

the line MC being then drawn, we can determine the point of application N of the resultant of the forces P+P′ applied at M, and of the force P″ applied at C; for we have

$$MC : MN :: P+P'+P'' : P'';$$

from which the value of MN results,

$$MN=\frac{MC\times P''}{P+P'+P''}.$$

By connecting the points N and D, the point of application O, of the four forces P, P′, P″, P‴, may be found in a manner precisely similar, and lastly, by joining the points O and E, we shall determine the point K at which the resultant of the entire system must be applied.

77. When some of the forces of which the system is composed exert their efforts in a contrary direction, we reduce the components P, P′, P″, &c., which are supposed to act in the same direction, to a single resultant equal to their sum, and likewise the components Q, Q′, Q″, &c., which are supposed to act in a contrary direction, to a second resultant equivalent to their sum; then, having determined the points of application K and L (*Fig.* 38) of these two resultants, the system will be reduced to two parallel forces, the one applied at K, and equal to P+P′+P″ &c., the other at L, and equal Q+Q′+Q″ &c.: the resultant of these two forces may then be determined by the method explained in Art. 74.

78. If the forces P, P′, P″, P‴, &c. (*Fig.* 39), retaining their points of application, and continuing parallel, assume the positions AQ, BQ′, CQ″, DQ‴, &c., the resultant will be parallel to the new directions of the forces, but its intensity and point of application will remain unchanged; for, the

construction employed to determine this resultant, being dependent only on the intensities of the forces and their points of application, the data of the problem will remain the same.

79. If, for example, the forces P and P′ should assume the positions represented by the parallels AQ and BQ′; there would be given P, P′, and the line AB, to determine the position of the point M; and this would be determined from the same proportion as when the forces were directed along the lines AP and BP′.

The point through which the resultant of a system of parallel forces constantly passes, whatever may be the direction of those forces, is called *the centre of parallel forces.*

80. To determine the co-ordinates of the centre of parallel forces, let P, P′, P″, &c. represent the intensities of the several forces, and denote by

x, y, z, the co-ordinates of the point of application M of the force P,
x', y', z' . . . . . . those of M′,
x'', y'', z'' . . . . . . those of M″,
. . . .
x_1, y_1, z_1, ... those of the centre of parallel forces.

If we represent by N (*Fig.* 40) the point of application of the resultant of the two forces P and P′, we shall have

$$MM' : M'N :: P+P' : P;$$

and by drawing the line ML′ parallel to HH′, the projection of MM′ on the plane of x, y, the similar triangles ML′M′, NLM′ will give

$$MM' : M'N :: ML' : NL;$$

whence, by combining the two proportions,

$$ML' : NL :: P+P' : P;$$

from which results the equation

$$(P+P')NL=P\times ML':$$

adding to each member the product (P+P′)LK, we have

$$(P+P')(NL+LK)=P(ML'+LK)+P'\times LK;$$

and since

$$NL+LK=NK,$$

$$ML'+LK=MH,$$
$$LK=M'H',$$

the preceding equation may be reduced to

$$(P+P')NK=P\times MH+P'\times M'H'.$$

If we denote by Q the resultant of the two forces P and P', and by Z the co-ordinate of its point of application, this equation may be written under the form

$$QZ=Pz+P'z';$$

in like manner, representing by Q' the resultant of the parallel forces Q and P'', and by Z' the co-ordinate of the point at which it is applied, we obtain

$$Q'Z'=QZ+P''z'';$$

and thence, by substitution,

$$Q'Z'=Pz+P'z'+P''z''.$$

If the resultant of the entire system be represented by R, and the co-ordinate of its point of application, parallel to the axis of z, by z_1, we shall obtain, by continuing the same process, the general relation

$$Rz_1=Pz+P'z'+P''z''+\&c. \ldots\ldots (36).$$

81. *The moment of a force with reference to a plane is the product of the intensity of this force by the distance of its point of application from the plane.* The preceding equation therefore expresses that *the moment of the resultant of the parallel forces* P, P', P'', *&c., taken with reference to the plane of x, y, is equal to the sum of the moments of the several forces taken with reference to the same plane.*

The moments being taken with reference to the other two co-ordinate planes, we have

$$Ry_1=Py+P'y'+P''y''+\&c. \ldots\ldots (37).$$
$$Rx_1=Px+P'x'+P''x''+\&c. \ldots\ldots (38).$$

82. When the co-ordinates x, y, z, x', y', z', &c. of the points of application, and the intensities P, P', P'', &c. of the forces, are given, the intensity of the resultant will become known, being equal to the algebraic sum of the several intensities; and the values of the co-ordinates x_1, y_1, and z_1, of the centre of parallel forces, will be found from the equations (36), (37), and (38).

83. The forces are affected with the positive or negative sign according to the directions in which they act; and since the signs of the co-ordinates are likewise determined by their positions with respect to the origin of co-ordinates, the moments of the forces must be regarded as positive, when the forces and co-ordinates have the same sign, but negative when the two have contrary signs.

84. If the several points of application M, M′, M″, &c. were situated in the same plane MM″ (*Fig.* 41), the plane of x, y might then be assumed parallel to that in which the forces are applied, and the co-ordinates z, z', z'', &c., being comprised between two parallel planes, we should have

$$z=z'=z''=\&c.:$$

hence, if $z_{,}$ represent the co-ordinate of the centre of parallel forces, its value will also be equal to z; for, its extremity must be found in the plane MM″, being determined by a construction similar to that in Art. 76. Thus the quantity z becomes a common factor in the equation (36) which then reduces to

$$\mathrm{R}=\mathrm{P}+\mathrm{P}'+\mathrm{P}''+\&c.$$

85. If the points of application were situated on the right line AB (*Fig.* 42), which we will suppose parallel to the axis of x, we should have at the same time

$$z=z'=z''=\&c., \text{ and } y=y'=y''=\&c.;$$

the equations (36) and (37) would then reduce to

$$\mathrm{R}=\mathrm{P}+\mathrm{P}'+\mathrm{P}''+\&c. \ldots\ldots (39),$$

and there would remain but the single equation

$$\mathrm{R}x_1=\mathrm{P}x+\mathrm{P}'x'+\mathrm{P}''x''+\&c. \ldots\ldots (40).$$

In this case, we may dispense with the consideration of the three axes, it being only necessary to estimate the co-ordinates x, x', x'', &c. along the line AB, to which the forces are applied.

For example, if we had

$$x=9, \quad x'=3, \quad x''=-3, \quad x'''=-4.$$

$$\mathrm{P}'=-\tfrac{1}{3}\mathrm{P}, \quad \mathrm{P}''=-\tfrac{2}{3}\mathrm{P}, \quad \mathrm{P}'''=2\mathrm{P};$$

by substituting these values in the equations (39) and (40) we should deduce

$$R=P-\tfrac{1}{3}P-\tfrac{2}{3}P+2P=2P,$$

$$Rx_1=9\times P-3\times\tfrac{1}{3}P+3\times\tfrac{2}{3}P-4\times 2P=1\times 2P;$$

whence,

$$x_1=1.$$

86. For the purpose of determining the conditions of equilibrium of parallel forces, we shall adopt as most convenient that position of the axes in which one of the co-ordinate planes is perpendicular to the direction of the forces: let this be the plane of x, y. Having reduced all the forces which act in the same direction to a single resultant $R_{\prime}$ (*Fig.* 43), and those which act in a contrary direction to a second resultant $R_{\prime\prime}$, an equilibrium will take place in the system when the two resultants are equal and directly opposed.

The latter condition will be fulfilled when the distance $C'C''$ is equal to zero, which requires that the co-ordinates $x_{\prime}$ and $y_{\prime}$ of the point C' should be respectively equal to $x_{\prime\prime}$ and $y_{\prime\prime}$ those of the point C''.

Hence, we obtain

$$x_{\prime}=x_{\prime\prime},\quad y_{\prime}=y_{\prime\prime}.$$

The condition of equality between the two resultants will be satisfied when we have

$$R_{\prime}=-R_{\prime\prime}\ \ldots\ldots\ (41);$$

and we obtain by multiplication

$$R_{\prime}x_{\prime}=-R_{\prime\prime}x_{\prime\prime}\ \ldots\ldots\ (42),$$

$$R_{\prime}y_{\prime}=-R_{\prime\prime}y_{\prime\prime}\ \ldots\ldots\ (43).$$

If we denote by P, P', P'', &c. the components of $R_{\prime}$, and by P''', P^{iv}, &c. the components of $R_{\prime\prime}$, the property of the moments will give the two equations

$$R_{\prime}x_{\prime}=Px+P'x'+P''x''+\&c.,$$

$$R_{\prime\prime}x_{\prime\prime}=P'''x'''+P^{iv}x^{iv}+P^{v}x^{v}+\&c.;$$

and substituting these values in equation (42), it reduces to

$$Px+P'x'+P''x''+P'''x'''+P^{iv}x^{iv}+P^{v}x^{v}+\&c.=0\ \ldots\ldots\ (44).$$

By the same course of reasoning, the equation (43) may be reduced to

$$Py+P'y'+P''y''+P'''y'''+P^{iv}y^{iv}+P^{v}y^{v}+\&c.=0\ \ldots\ldots\ (45).$$

And finally, the values of $R_{\prime}$ and $R_{\prime\prime}$ being substituted in equation (41), give

$$P+P'+P''+P'''+P^{\text{iv}}+P^{\text{v}}+\&c.=0 \ldots\ldots (46).$$

87. If the equations (44), (45), and (46) are satisfied, the system of forces will be in equilibrio. The conditions expressed by these equations may be enunciated as follows: *An equilibrium will subsist in a system of parallel forces, if the sum of the moments taken with reference to each of two rectangular planes parallel to the common direction of the forces, is equal to zero; the sum of the forces being at the same time equal to zero.*

88. An equilibrium will also take place if the resultant of the system be supposed to pass through a fixed point, since the effect of this resultant will then be destroyed by the resistance opposed by the fixed point.

Of Forces situated in the same Plane, and applied to Points connected together in an invariable manner.

89. Let P, P′, P″, P‴, &c. (*Fig.* 44) represent several forces situated in the same plane, and applied to the points A, B, C, D, &c., which are supposed to be connected in an invariable manner. If the system admits of a single resultant, its position and intensity may be readily obtained by means of the following graphic construction:—Having assumed the portions A*a*, B*b*, C*c*, and D*d* proportional to the intensities of the respective forces, prolong the lines A*a* and B*b* until they intersect at the point G, and apply the forces P and P′ at this point. Construct the parallelogram GG′, having its sides respectively equal to A*a* and B*b*, and its diagonal GG′ will represent in direction and intensity the resultant of the two forces P and P′; again, by prolonging GG′ and C*c* until they intersect, and constructing the parallelogram HH′, whose sides shall represent the forces GG′ and C*c*, the diagonal HH′ will represent the resultant of these forces, and will therefore be the resultant of the three forces P, P′, and P″. Lastly, by finding the intersection of HH′ and D*d*, and forming a third

parallelogram, its diagonal II′ will represent the resultant of the entire system.

90. If by this construction we should find one or more pairs of parallel forces, the resultant may be determined by the methods explained in Arts. (71), (72), and (74), and its intensity will be equal to the sum or difference of the forces. If the system contain two parallel and equal forces, acting in contrary directions, but not directly opposed, we may combine one of them with the other forces, and the construction of Art. (89) may then be continued; but if the entire system can be reduced to two equal resultants acting in parallel and contrary directions, but not directly opposed, we conclude, as in Art. 74, that a single resultant cannot be obtained.

91. If the construction should give a resultant equal to zero, an equilibrium would subsist throughout the system.

92. The preceding construction is equivalent to supposing the forces applied at the point I, in lines parallel to their primitive directions, and then compounding them into a single resultant. For, by considering the forces P, Q, and S (*Fig.* 45), the resultant DC of the forces P and Q, being applied at the point D′ in its line of direction, may there be decomposed into the two components D′P′ and D′Q′, parallel and equal to P and Q.

93. To determine the analytical conditions of equilibrium in a system of forces disposed like the preceding, we will first consider the case of three forces P, P′, and P″, applied to points which are connected in an invariable manner; and we shall then find it necessary that the directions of the forces should intersect in a single point. For, since the forces P and P′ (*Fig.* 46) are supposed to be sustained in equilibrio by the third force P″, it is necessary that this third force should be equal and directly opposed to the resultant of the two forces P and P′. But P and P′ intersect in a point D; this point is therefore situated on their resultant, and consequently in the direction of the third force P″.

If, on the contrary, the force P″ were not applied at the point of intersection of the other two, it would intersect the direction of their resultant R at some point E (*Fig.* 47), and the right lines RD and P″E being then inclined to each other in a certain angle P″ER, the forces R and P″ could not maintain an equilibrium (Art. 16).

94. When the directions of the three forces P, P′, P″ intersect in a point, this point may be considered as their point of application, and the conditions of equilibrium will then be the same as if the forces had been originally applied at their point of intersection.

These conditions are,

$$P \cos \alpha + P' \cos \alpha' + P'' \cos \alpha'' + \&c. = 0,$$
$$P \cos \beta + P' \cos \beta' + P'' \cos \beta'' + \&c. = 0.$$

To these must be added the equation which expresses the condition of their intersecting in a point.

95. Let P, P′, and R (*Fig.* 48) represent three forces whose directions intersect at the point A. If through the point C, assumed arbitrarily, a right line be drawn to the point A, and perpendiculars CI, CI′, CI″ be demitted on the lines of direction of the forces, the right-angled triangles CAI, CAI′, CAI″ will have the same hypotheneuse CA: this condition of a common hypotheneuse will establish that of the forces intersecting at a single point, since it results from the triangles having a common vertex. Through the point A draw the right line AB, perpendicular to CA, and from the extremities of the lines AP, AP′, and AR, which represent the intensities of the forces, demit perpendiculars PD, P′D′, RD″ on the line AB: the right-angled triangles ACI and APD will be similar, having the alternate angles CAI and APD equal to each other, and the following proportion will therefore obtain:

$$AC : CI :: AP : AD,$$

and by calling $AC = c$, $CI = p$, this proportion becomes

$$c : p :: P : AD;$$

whence we obtain

$$AD = \frac{Pp}{c}:$$

denoting by p' and r the perpendiculars CI′ and CI″, we find, in like manner,

$$AD' = \frac{P'p'}{c}, \quad AD'' = \frac{Rr}{c}.$$

But if R be the resultant of P and P′, the component of R in the direction of AB will be equal to the sum of the components of P and P′, directed along the same line; we consequently have

$$AD''=AD+AD';$$

and by substituting in this equation the values found above, it becomes

$$\frac{Rr}{c}=\frac{Pp}{c}+\frac{P'p'}{c};$$

or, by suppressing the divisor common to the terms, it reduces to

$$Rr=Pp+P'p' \ldots\ldots (47).$$

96. If the point C were situated within the angle formed by the directions of the forces, or in the opposite angle, the product of the resultant by the perpendicular r would then be equal to the difference of the products of the two components multiplied by their respective perpendiculars; we should thus have

$$Rr=Pp-P'p' \ldots\ldots (48).$$

97. The moment of a force with reference to a plane has been defined (Art. 81) to be the product of the intensity of this force by the perpendicular on the plane from the point of application. By analogy, we call *the moment of a force with reference to a point, the product of the force by the perpendicular demitted on the direction of the force from the assumed point.* The equations (47) and (48) will therefore express that the moment of the resultant of two forces is equal to the sum or difference of the moments of its components, according to the position of the point C. This point is called *the centre of moments;* and if it be situated within the angle PAP′, or LAL′ (*Fig.* 49), the difference of the moments must be taken, but if it fall without these angles, the moment of the resultant will be equal to the sum of the moments.

98. These two cases may be comprised in a single enunciation, by attaching to the word sum its algebraic signification, and the moment of the resultant will then be equal to the sum of the moments of the two components, in which expression the terms may be affected either with the positive or negative signs.

99. **The condition of the forces intersecting in a point gives rise to the preceding theorem of the moments: from this theorem the third condition of equilibrium may be deduced.**

For, if two forces P and P′ (*Fig.* 50) are sustained in equilibrio by a third force P″, this force must be equal in intensity to the resultant of the other two, and must act in a direction exactly opposite. If, therefore, a perpendicular p'' be demitted on the line of direction of the force P″, which is also that of the resultant R, the principle of the moments will furnish the equation

$$Rp''=Pp+P'p';$$

and replacing R by —P″, since the forces are equal, and act in contrary directions, the equation becomes

$$Pp+P'p'+P''p''=0.$$

Thus the conditions of equilibrium of three forces situated in the same plane, and applied to different points, will be expressed by the three following equations :—

$$P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''=0\ \ldots\ldots\ (49),$$

$$P\cos\beta+P'\cos\beta'+P''\cos\beta''=0\ \ldots\ldots\ (50),$$

$$Pp+P'p'+P''p''=0\ \ldots\ldots\ (51).$$

100. If the number of forces be greater than three, we may regard P as being the resultant of the two forces P‴ and P$^{\text{iv}}$: we shall then have

$$P\cos\alpha=P'''\cos\alpha'''+P^{\text{iv}}\cos\alpha^{\text{iv}},$$

$$P\cos\beta=P'''\cos\beta'''+P^{\text{iv}}\cos\beta^{\text{iv}},$$

$$Pp=P'''p'''+P^{\text{iv}}p^{\text{iv}};$$

and by substituting these values in equations (49), (50), (51), they become

$$P'\cos\alpha'+P''\cos\alpha''+P'''\cos\alpha'''+P^{\text{iv}}\cos\alpha^{\text{iv}}=0,$$

$$P'\cos\beta'+P''\cos\beta''+P'''\cos\beta'''+P^{\text{iv}}\cos\beta^{\text{iv}}=0.$$

$$P'p'+P''p''+P'''p'''+P^{\text{iv}}p^{\text{iv}}=0.$$

101. The same principle may be extended to any number of forces, and we shall therefore obtain for the general equations of equilibrium of forces acting in the same plane, and applied to different points,

$$P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''+\&c.=0\ \ldots\ldots\ (52),$$

$$P\cos\beta+P'\cos\beta'+P''\cos\beta''+\&c.=0\ \ldots\ldots\ (53),$$

$$Pp+P'p'+P''p''+\&c.=0\ \ldots\ldots\ (54).$$

102. A more convenient notation is sometimes employed to express the existence of these conditions, the equations being written in the following form :—

$$\Sigma(P\cos\alpha)=0,\quad \Sigma(P\cos\beta)=0,\quad \Sigma(Pp)=0.$$

The character Σ is here employed to denote the sum of any number of quantities of the same form as those included within the parentheses.

103. The process which has led to the equation (47) furnishes an easy method of recognising the proper signs of the moments. For, if the point C, the centre of moments (*Fig.* 51), be chosen without the angle formed by the directions of the extreme forces, and the forces be supposed to act by pushing, being at the same time firmly connected with the perpendiculars p, p', p'', &c., these forces will all tend to turn the perpendiculars in the same direction about the point C; but if, on the contrary, the centre C be situated within the angle formed by the directions of the extreme forces (*Fig.* 52), or within the opposite angle, the forces P, P′, P″, &c., situated on the same side of the point C, will tend to turn the perpendiculars in one direction, while the forces P‴, P^iv, &c., on the opposite side, will tend to turn the perpendiculars in a contrary direction. But the expressions $\frac{Pp}{c}$, $\frac{P'p'}{c}$, $\frac{P''p''}{c}$, &c., represented by the lines AD, AD′, AD″, &c., being affected with signs contrary to those of AD‴, AD^iv, &c., it follows that all the forces whose moments are positive will tend to turn the system in one direction, while those whose moments are negative will tend to turn it in a contrary direction.*

---

* This demonstration is perfectly conclusive when the directions of the several forces intersect in a point; but the property of the moments is equally true when the forces are not directed to a single point. For, by prolonging the directions of any two of the forces P and P′ until they intersect, and joining their point of intersection with the centre of moments, it may be proved by the reasoning employed in Art. 108, that the moment of their resultant is equal to the algebraic sum of the moments of the two forces P and P′, the signs of these moments being determined by the directions in which the forces P and P′ tend to turn the system about the centre of moments. We shall thus have

104. If the system of forces be not in equilibrio, the moment of the resultant will be equal to the excess of the sum of the moments of those forces which tend to produce rotation in one direction, over the sum of the moments of those which tend to turn the system in a contrary direction.

105. It appears from the preceding remarks, that the equation $\Sigma(Pp)=0$, expresses the condition that the sums of the moments of the forces which tend to produce rotation in the two directions are equal to each other.

106. If, in the system supposed in equilibrio, we suppress one of the components, P for example, the remaining forces will have a resultant R; and since this resultant should be equal in intensity, but directly opposed to the force P, the equations (52), (53), and (54) will be replaced by the following:

$$R\cos a = P'\cos\alpha' + P''\cos\alpha'' + P'''\cos\alpha''' + \&c.,$$
$$R\cos b = P'\cos\beta' + P''\cos\beta'' + P'''\cos\beta''' + \&c.,$$
$$Rr = P'p' + P''p'' + P'''p''' + \&c.;$$

or,

$$R\cos a = \Sigma(P\cos\alpha) = X,$$
$$R\cos b = \Sigma(P\cos\beta) = Y,$$
$$Rr = \Sigma(Pp).$$

$$Rr = Pp \pm P'p'.$$

The double sign is not prefixed to the moment Pp, since we are at liberty to assume arbitrarily the sign of one of the moments. The moment Rr, deduced from this equation, may have either a positive or negative value; if positive, R and P will tend to turn the system in the same direction; if negative, in contrary directions.

The forces P and P′, being then replaced by their resultant R, this resultant can be combined with a third force P″, and we shall obtain, in a similar manner,

$$R'r' = Rr \pm P''p'';$$

in which equation Rr, whatever may be its essential sign, may be replaced by $Pp \pm P'p'$. The sign of the moment $P''p''$ will be similar to that of Rr, if P″ and R tend to produce rotation in the same direction, and dissimilar in the contrary case. But the moments Pp and Rr will have like or unlike signs, according as the forces P and R tend to turn the system in the same or in contrary directions. Hence the signs of the moments Pp and $P''p''$ in the equation $R'r' = Pp \pm P'p' \pm P''p''$, will be like or unlike according to the directions in which the forces P and P″ tend to produce rotation.

The same reasoning may be extended to a greater number of forces.

107. By means of these equations, the position and magnitude of the resultant may be determined.

For, the two first equations give

$$R^2(\cos^2 a+\cos^2 b)=X^2+Y^2;$$

and since the sum of the squares of the two cosines is equal to unity, we have

$$R^2=X^2+Y^2.$$

The inclinations of the resultant to the co-ordinate axes may also be determined from the same equations; for we have

$$\cos a=\frac{X}{R}, \quad \cos b=\frac{Y}{R}.$$

108. To establish its position in the system, we first determine the position of a right line AB, passing through the origin, and parallel to the resultant. If cos b be affected with the positive sign, the line AB must form with the axis of y an angle less than 90°: it will therefore assume one of the positions indicated in (*Fig.* 53). But if, on the contrary, this quantity should have the negative sign, the right line AB would then be situated in one of the positions represented by (*Fig.* 54). Thus, whatever be the sign of cos b, the line AB may assume two positions, one in which the angle formed with Ax will be obtuse, and another in which this angle will be acute. The sign of the cos a will determine which of these positions the line AB must assume.

Having thus established the position of the right line AB, let a perpendicular r be drawn to it through the origin A, equal to $\frac{\Sigma(Pp)}{R}$. This perpendicular will be represented (*Fig.* 55) by AO or by AO′, according to the sign of the quantity r; and the line OR or O′R′, parallel to AB, will represent the true position of the resultant.

109. To obtain the equation of this resultant, it may be observed that its line of direction will, in general, intersect the axis of y at a certain point B (*Fig.* 56), and that the form of its equation will therefore be

$$y=x \text{ tang } D+AB \ldots\ldots (55);$$

and since the angle which the resultant makes with the axis of x is denoted by a, we have $D=a$, and consequently

$$\text{tang } D=\frac{\sin a}{\cos a}=\frac{\cos b}{\cos a}=\frac{R\cos b}{R\cos a}=\frac{Y}{X}.$$

The value of AB may be obtained from the equation

$$OA=AB\times\cos OAB.$$

But the angle OAB is equal to the angle D, since they are both complements of OAD. The angle OAB can therefore be replaced in the preceding equation by D or a; and since the line AO is the perpendicular from the origin on the direction of the resultant, it will represent the quantity denoted by r; we shall thus obtain

$$r=AB\cos a;$$

and consequently,

$$AB=\frac{r}{\cos a}.$$

Substituting the values of AB and tang D in the general equation (55), it becomes

$$y=\frac{Y}{X}x+\frac{r}{\cos a}=\frac{Y}{X}x+\frac{Rr}{R\cos a}=\frac{Y}{X}x+\frac{Rr}{X};$$

whence, by transposition and reduction, we find

$$yX-xY=Rr;$$

or, replacing Rr by its equal $\Sigma(Pp)$, the equation of the resultant finally becomes

$$yX-xY=\Sigma(Pp).$$

110. When an equilibrium subsists, X and Y are equal to zero, and the equation reduces to $\Sigma(Pp)=0$, corresponding with the result previously obtained.

111. The data requisite for the determination of the resultant being, 1°. The intensities of the several forces; 2°. The angles on which their directions depend; and 3°. The co-ordinates of their points of application, it will prove convenient to transform the equation (54) into another, in which the quantities p, p', p'', &c. shall be replaced by the co-ordinates of the points of application. To effect this transformation, let the origin of co-ordinates be assumed at A (*Fig.* 57),

and let x and y denote the co-ordinates of the point M to which a force P is applied: the intensity of this force being represented by MP, its components parallel to the axes of x and y will be respectively

$$\mathrm{MN}=\mathrm{P}\cos\alpha,$$
$$\mathrm{MQ}=\mathrm{P}\cos\beta.$$

From the point A demit the perpendiculars AO, AF, and AE on the prolongations of the force MP and its two components; we shall then have

$\mathrm{OA}\times\mathrm{MP}=$ the moment of the force P,

$\mathrm{AF}\times\mathrm{MN}=$ the moment of the component $\mathrm{P}\cos\alpha$,

$\mathrm{AE}\times\mathrm{MQ}=$ the moment of the component $\mathrm{P}\cos\beta$.

But if we regard the forces as pushing the point M, the resultant MP and the component $\mathrm{P}\cos\alpha$ will tend to produce rotation in the same direction about the point A. Their moments may therefore be affected with the positive sign; while the component $\mathrm{P}\cos\beta$, tending to turn the system in a contrary direction, must be affected with the negative sign. We shall thus obtain the equation

$$\mathrm{P}p=y\mathrm{P}\cos\alpha-x\mathrm{P}\cos\beta.$$

For a similar reason,

$$\mathrm{P}'p'=y'\mathrm{P}'\cos\alpha'-x'\mathrm{P}'\cos\beta',$$
$$\mathrm{P}''p''=y''\mathrm{P}''\cos\alpha''-x''\mathrm{P}''\cos\beta'',$$
$$\&c. \qquad \&c. \qquad \&c.;$$

and by substituting these values in the equation of the moments (54), it becomes

$$\mathrm{P}(y\cos\alpha-x\cos\beta)+\mathrm{P}'(y'\cos\alpha'-x'\cos\beta')+\&c.=0\ \ldots.\ (56):$$

we shall therefore have for the equation of the resultant, when the system is not in equilibrio (Art. 109),

$$y\mathrm{X}-x\mathrm{Y}=\Sigma[\mathrm{P}(y\cos\alpha-x\cos\beta)].$$

112. In determining the signs of the moments in equation (54), we had recourse to the rule explained in Art. 108, which is somewhat foreign to analytical considerations; but when, by a transformation, this equation takes the form indicated above (56), the signs of the moments will be immediately determined by an application of the rule in Arts. 37 and 38, regard being had to the signs of the co-ordinates. Thus, let

P be a force whose position with respect to the co-ordinate axes is that represented in (*Fig.* 58). The value of its moment, being in general $P(y \cos \alpha - x \cos \beta)$, will become applicable to the particular case, by making x negative, y positive, $\cos \alpha$ negative, $\cos \beta$ negative: thus, when the signs are considered, the moment becomes

$$P(-y \cos \alpha - x \cos \beta).$$

113. It should be remarked, however, that we here adopt tacitly an hypothesis relative to the signs, which consists in regarding a moment as positive, when the direction of the force CD (*Fig.* 57) intersects the axis of y positive, and then cuts the axis of x negative.

114. The equations of equilibrium (49), (50), and (51) imply the condition that the system may be reduced to two forces equal in intensity and directly opposite. For, if we denote by $P \cos \alpha$, $P' \cos \alpha'$, &c. the components acting in one direction parallel to the axis of x, and by $P'' \cos \alpha''$, $P''' \cos \alpha'''$, &c. the components which act in a contrary direction, the equation (49) may be put under the form

$$P \cos \alpha + P' \cos \alpha' + \&c. = P'' \cos \alpha'' + P''' \cos \alpha''' + \&c.$$

But the forces $P \cos \alpha$, $P' \cos \alpha'$, &c., being parallel, may be compounded into a single force X', equal to their sum and parallel to them; and the forces $P'' \cos \alpha''$, $P''' \cos \alpha'''$, &c. may in like manner be replaced by a single force X'': the entire system will thus be reduced to the two forces X' and X'', parallel and equal, but having contrary directions.

By a similar composition, the forces parallel to the axis of y may be reduced to two resultants Y' and Y'', equal to each other, and having opposite directions.

The forces X' and Y' being then applied at the point M, where their directions intersect (*Fig.* 59), and the forces X'' and Y'' at their point of intersection N, we can construct the rectangles MA and NB, whose sides MC, MD, NE, and NF shall represent the forces X', Y', X'', Y'': and since the homologous sides of these rectangles are equal, their diagonals MA and NB will also be equal and parallel.

The equations $X=0$, $Y=0$, therefore, express that forces situated in a plane may be reduced to two MA and NB, equal, parallel, and acting in contrary directions; but they do not express the condition that the two forces are directly opposed. That this may occur, the equation $\Sigma(Pp)=0$ is likewise necessary: for, calling R' and R'' the two equal forces AM and BN, and r', r'' the perpendiculars OP and OQ demitted from the point O, since R' and R'' act in contrary directions, their moments must be taken with different signs, and the equation $\Sigma(Pp)=0$ will be replaced by the following:

$$R'r'-R''r''=0.$$

But the intensities of R' and R'' being equal by hypothesis, the common factor will disappear from the equation, and it will then become

$$r'-r''=0;$$

thus, the difference of the right lines OP and OQ will become equal to zero, and the points P and Q will therefore coincide: hence, the forces MA and NB will be directed along the same right line.

It also appears that when the condition $\Sigma(Pp)=0$ is not fulfilled, and we have simply $X=0$, $Y=0$, the system may be reduced to two parallel forces similarly situated to those considered in Art. 74.

115. If, on the contrary, the condition $\Sigma(Pp)=0$ were alone satisfied, an equilibrium could not subsist; for the quantities X and Y having certain values, a resultant might be found whose intensity would be determined by means of the equation

$$R=\surd(X^2+Y^2).$$

In this case, the equation $\Sigma(Pp)=0$, or its equivalent $Rr=0$, can only be satisfied by making the factor r equal to zero; hence, the centre of moments must necessarily be found on the line of direction of the resultant R.

116. If there be a fixed point on the line of direction of the resultant, the equilibrium will be still maintained, and the centre of moments being placed at this point, the condition $\Sigma(Pp)=0$ will be satisfied; if, for example, the forces P, P′, P″, &c. be supposed applied to the different points of a solid body, and if the point C through which the resultant passes be immoveable, the effect of this resultant will be entirely destroyed

by the reaction of the fixed point, and the condition $\Sigma(Pp)=0$ will be alone sufficient to ensure the equilibrium. It will appear hereafter that the intensity of this resultant is a measure of the pressure sustained by the fixed point.

117. If the system can be reduced to two parallel forces, equal in intensity, but not directly opposed, the addition of an arbitrary force S will render it susceptible of a single resultant. For the new force S must necessarily be either parallel or inclined to the direction of the forces; in the first case (*Fig.* 60), it may be decomposed into two parallel components P′ and Q′ applied at the points A and B (Art. 73), and the system of three forces P, Q, and S will be replaced by the two unequal forces P+P′ applied at A, and Q—Q′ applied at B; these two forces will obviously have a single resultant.

If the new force S is not parallel to the other two, its direction may be prolonged (*Fig.* 61) until it intersects the direction of one of them at A′. This point being then taken as the point of application of the forces P and S, they may be compounded by constructing a parallelogram on their lines of direction, and the direction of their resultant will intersect that of the force Q, with which force this resultant may be combined.

Of Forces acting in any manner in Space.

118. Let P', P'', P''', &c. represent different forces situated in space;

x', y', z', the co-ordinates of the point of application of P',

x'', y'', z'', those of P'',

x''', y''', z''', those of P''',

&c. &c. &c.;

α', β', γ', the angles formed by P' with the axes of co-ordinates,

$\alpha'', \beta'', \gamma''$, those formed by P'' with the axes,

$\alpha''', \beta''', \gamma'''$, those formed by P''' with the axes,

&c. &c. &c.

Let us investigate the conditions of equilibrium in this system, and endeavour to discover if these conditions cannot be rendered dependent on those which have been obtained in the preceding cases. We first attempt to decompose all the forces of the system into two groups, one of which shall consist of parallel components, and the second of forces situated in the same plane. Since the axes of co-ordinates may be assumed arbitrarily, we will endeavour to decompose the forces in such manner that a certain number of them may be in the plane of x, y, and the remainder be parallel to the axis of z.

119. If in the given system there be no force parallel to the plane of x, y, the proposed decomposition may be readily effected; for, let one of the forces be represented by P′, its point of application being at M′ (*Fig*. 62); prolong the line of direction of this force until it intersects at C′ the plane of x, y, and transferring the point of application to C′, decompose the force P′ into two others, one C′L parallel to the axis of z, the other C′N in the plane of x, y.

120. But if the force P′ is parallel to the plane of x, y, a similar decomposition cannot be effected, and some other mode of decomposing the forces must therefore be adopted.

For this purpose, let there be drawn through the point M′ (*Fig*. 63) a line parallel to the axis of z, and to the point M′ let there be applied along this line, and in contrary directions, the two forces M′O and M′O′, having intensities equal to g' and $-g'$ respectively. The introduction of these forces cannot disturb the condition of the system, since the two mutually destroy each other; and we shall then have applied at the point M′ the three forces P′, g', and $-g'$.

The force P′ may then be compounded with $-g'$, and by calling their resultant R′, we can replace in the system the force P′, by the two forces R′ and g', each of which must obviously intersect the plane of x, y.

121. Let the force R′ be now applied at C′, the point in which its line of direction intersects the plane of x, y, and let it be decomposed into two components, one situated in the plane of x, y, and the other parallel to the axis of z. The force P′ will thus be replaced by a force applied at C′, and lying in the plane of x, y, and by two others parallel to the axis of z, one applied at C′, and the other at M′.

122. The co-ordinates of the points of application being

necessary to express the conditions of equilibrium, those of the point C′ must be determined.

The equations of the resultant R′ which passes through the point x', y', z', have been found (Art. 57) to be of the form

$$\left.\begin{aligned} z-z' &= \frac{Z}{X}(x-x') \\ z-z' &= \frac{Z}{Y}(y-y') \end{aligned}\right\} \ldots\ldots (57);$$

in which X, Y, and Z represent the projections of R′ on the co-ordinate axes. These projections being equal to the components of R′ parallel to the axes, the quantities X, Y, and Z may be replaced by the values of the three components. But R′ being the resultant of P′ and $-g'$, we may substitute for P′ its three components $P'\cos\alpha'$, $P'\cos\beta'$, $P'\cos\gamma'$; and R′ will then be the resultant of the four forces

$$P'\cos\alpha',\quad P'\cos\beta',\quad P'\cos\gamma',\quad -g'.$$

These forces acting parallel to the axes of co-ordinates, we shall have

$$X=P'\cos\alpha',\quad Y=P'\cos\beta',\quad Z=P'\cos\gamma'-g';$$

and by substituting these values in equations (57), we obtain for the equations of the resultant R′,

$$\left.\begin{aligned} z-z' &= \frac{P'\cos\gamma'-g'}{P'\cos\alpha'}(x-x') \\ z-z' &= \frac{P'\cos\gamma'-g'}{P'\cos\beta'}(y-y') \end{aligned}\right\} \ldots\ldots (58).$$

123. To obtain the co-ordinates of the point C′ (*Fig.* 63), at which the right line R′ intersects the plane of x, y, we make $z=0$ in the equations (58); and denoting by $a_{,}$ and $b_{,}$ the other two co-ordinates of the point C′, we shall have

$$-z' = \frac{P'\cos\gamma'-g'}{P'\cos\alpha'}(a_{,}-x'),$$

$$-z = \frac{P'\cos\gamma'-g'}{P'\cos\beta'}(b_{,}-y');$$

from which we deduce

$$\left.\begin{aligned} a_{,} &= x' - \frac{z'P'\cos\alpha'}{P'\cos\gamma'-g'} \\ b_{,} &= y' - \frac{z'P'\cos\beta'}{P'\cos\gamma'-g'} \end{aligned}\right\} \ldots\ldots (59):$$

these are the values of the co-ordinates of the point C′, at which the resultant R′ intersects the plane of x, y.

124. The force R′, being represented in intensity by the line M′R′ (*Fig.* 64), may be supposed applied at C′, in its line of direction. Then making C′D′=M′R′, and decomposing C′D′ into three rectangular forces, applied at C′ and parallel to the co-ordinate axes, these components will be equal to those of the force M′R′; and the point C′ may therefore be considered as solicited by the three forces $P' \cos \alpha'$, $P' \cos \beta'$, and $P' \cos \gamma' - g'$, the two former being situated in the plane of x, y, and the latter parallel to the axis of z. Thus, instead of the force P′ applied at M′, we shall have

the force g' applied at M′, parallel to the axis of z,
the force $P' \cos \gamma' - g'$ applied at C′, parallel to the axis of z,
the force $P' \cos \alpha'$ applied at C′, and acting in the plane of x, y,
the force $P' \cos \beta'$ applied at C′, and acting in the plane of x, y.

125. By adopting a similar method of decomposition for the forces P″, P‴, &c., employing the auxiliary forces g'', g''', &c., applied at the points M″, M‴, &c., the system will be reduced to two groups of forces, of which one will have its components parallel to the axis of z, and the other will be situated in the plane of x, y.

The forces parallel to the axis of z will be

$$g',\, g'',\, g''', \text{ \&c.},$$

applied at the points M′, M″, M‴, &c.; and

$$P' \cos \gamma' - g', \quad P'' \cos \gamma'' - g'', \quad P''' \cos \gamma''' - g''', \text{ \&c.},$$

applied at the points C′, C″, C‴, &c.

And the forces lying in the plane of x, y, will be

$$P' \cos \alpha', \quad P'' \cos \alpha'', \quad P''' \cos \alpha''', \text{ \&c.},$$

applied at the points C′, C″, C‴, &c.; and

$$P' \cos \beta', \quad P'' \cos \beta'', \quad P''' \cos \beta''', \text{ \&c.}$$

applied at the same points C′, C″, C‴, &c.

126. It will now be demonstrated that when an equilibrium subsists in the system, it will be necessary, 1°. that the forces parallel to the axis of z should be in equilibrio; 2°. that the forces acting in the plane of x, y, should also destroy each other.

For since the equilibrium is supposed to subsist, the state of the system will not be changed by supposing a line C′C″ assumed arbitrarily in the plane of x, y (*Fig.* 65), to become immoveable. The forces situated in this plane will then be destroyed by the resistance of the fixed line. For, every force in the plane of x, y must intersect the fixed line, or be parallel to it. In the first case, let the force be represented by AB, and prolong its line of direction until it intersects the fixed line at a point O: this point being supposed immoveable, the effect of the force AB, which is transmitted to the point, must be destroyed. Again, if the force be parallel to the line C′C″, its point of application E cannot be moved without communicating a motion to the line C′C″ which by hypothesis is immoveable. The effect of this force must therefore be destroyed by the fixed line. Thus, the forces lying in the plane of x, y being destroyed, the system will be reduced to the group parallel to the axis of z. These latter forces would obviously tend to turn the system about the fixed line C″C′, unless the forces should be in equilibrio, or their resultant should pass through the fixed line. But the position of this line having been assumed arbitrarily, it cannot happen that the resultant of the forces parallel to the axis of z will always pass through this line. These parallel forces must therefore be in equilibrio.

The group parallel to the axis of z being in equilibrio, the forces lying in the plane of x, y must mutually destroy each other, since the equilibrium of the entire system could not otherwise be preserved.

127. The problem is thus reduced to finding the conditions of equilibrium, 1°. of a system of forces parallel to the axis of z; 2°. of the forces acting in the plane of x, y.

Conditions of Equilibrium of the Forces parallel to the Axis of z.

128. These conditions being the same as those enunciated in Art. 87, the following quantities must be equal to zero,—

1°. The sum of the forces parallel to the axis of z;

2°. The sum of the moments taken with reference to the plane of y, z;

3°. The sum of the moments taken with reference to the plane of x, z.

The first of these conditions gives

$$\begin{gathered}P'\cos\gamma'-g'+g'+P''\cos\gamma''-g''+g''\\+P'''\cos\gamma'''-g'''+g'''+\&c.=0;\end{gathered}$$

or, by reduction,

$$P'\cos\gamma'+P''\cos\gamma''+P'''\cos\gamma'''+\&c.=0\ \ldots\ldots\ (60).$$

The second condition requires the consideration of two different sets of moments.

1°. Those of the forces g', g'', g''', &c., applied at the points M′, M″, M‴, &c.

2°. Those of the forces $P'\cos\gamma'-g'$, $P''\cos\gamma''-g''$, $P'''\cos\gamma'''-g'''$, &c., applied at the points C′, C″, C‴, &c.

The moment of the force g' applied at M′ (*Fig.* 66), taken with reference to the plane of y, z, is $g'\times MN'$: but $MN'=B'D'=x'$; the moment therefore becomes $g'x'$.

The moment of the force $P'\cos\gamma'-g'$ applied at C′, taken with reference to the same plane, is evidently $(P'\cos\gamma'-g')\times E'C'$, or $(P'\cos\gamma'-g')a_{,}$; and the sum of the moments of the two forces will therefore be represented by

$$g'x'+(P'\cos\gamma'-g')a_{,}.$$

Substituting in this expression the value of $a_{,}$ (59) determined in Art. 123, we obtain

$$g'x'+(P'\cos\gamma'-g')\left(x'-\frac{zP'\cos\alpha'}{P'\cos\gamma'-g'}\right);$$

performing the multiplications indicated, and reducing, we get

$$x'P'\cos\gamma'-z'P'\cos\alpha'.$$

By a similar process, the moments of the parallel forces applied at M″, M‴, C″, C‴, &c. may be obtained, and being collected into one sum, the equation expressing the second condition of equilibrium becomes

$$\begin{gathered}P'(x'\cos\gamma'-z'\cos\alpha')+P''(x''\cos\gamma''-z''\cos\alpha'')\\+P'''(x'''\cos\gamma'''-z'''\cos\alpha''')+\&c.=0\ \ldots\ldots\ (61).\end{gathered}$$

To obtain the third condition of equilibrium of parallel forces, we find the moment of the force g' applied at M′, taken with reference to the plane of x, z, and that of the force $P'\cos\gamma'-g'$ applied at C′, taken with reference to the same plane: the first of these will be equal to $g'\times M'L'=g'\times B'G'=g'\times y'$; the second will be $(P'\cos\gamma'-g')b_{,}$; and their sum will be expressed by

$$g'y'+(P'\cos\gamma'-g')b_{,}.$$

Substituting for $b_{,}$ its value (59) found in Art. 123, and reducing, we obtain

$$y'P'\cos\gamma'-z'P'\cos\beta'.$$

And by finding the moments of the other parallel forces, taken with reference to the plane of x, z, we shall have for the third condition of equilibrium,

$$P'(y'\cos\gamma'-z'\cos\beta')+P''(y''\cos\gamma''-z''\cos\beta'')$$
$$+P'''(y'''\cos\gamma'''-z'''\cos\beta''')+\&c.=0\,\ldots\ldots\,(62).$$

Conditions of Equilibrium of the Forces situated in the Plane of x, y.

129. These conditions being such as arise when the forces act in the same plane, it is necessary,

1°. That the sum of the components parallel to the axis of x should be equal to zero.

2°. That the sum of the components parallel to the axis of y should be equal to zero.

3°. That the sum of the moments of the forces taken with reference to the origin should be equal to zero.

The first two conditions are expressed by the equations,

$$P'\cos\alpha'+P''\cos\alpha''+P'''\cos\alpha'''+\&c.=0\,\ldots\ldots\,(63),$$

$$P'\cos\beta'+P''\cos\beta''+P'''\cos\beta'''+\&c.=0\,\ldots\ldots\,(64).$$

With regard to the third, it may be observed, that the two forces $P'\cos\alpha'$ and $P'\cos\beta'$ are applied at the point C′ (*Fig.* 67); the moment of the first, being taken with reference to the origin A, will be

$$P'\cos\alpha'\times AE'=P'\cos\alpha'\times C'F'=P'\cos\alpha'\,.\,b_{,};$$

in like manner, the moment of the force $P'\cos\beta'$, taken with reference to the origin A, will be

$$P'\cos\beta'\times AF'=P'\cos\beta'\times E'C'=P'\cos\beta'\,.\,a_{,}.$$

These moments should be taken with contrary signs, since the two components $P' \cos \alpha'$ and $P' \cos \beta'$ tend to turn the system in contrary directions about the point A. Thus, by regarding that moment as positive in which the component $P' \cos \alpha'$ enters, the sum of the moments may be written

$$P' \cos \alpha' \times b_{,} - P' \cos \beta' \times a_{,};$$

substituting in this expression the values of $a_{,}$ and $b_{,}$ (59), we get

$$P' \cos \alpha' \left(y' - \frac{z'P' \cos \beta'}{P' \cos \gamma' - g'}\right) - P' \cos \beta' \left(x' - \frac{z'P' \cos \alpha'}{P' \cos \gamma' - g'}\right);$$

and by performing the multiplications, and reducing, we obtain

$$y'P' \cos \alpha' - x'P' \cos \beta'.$$

The moments of the forces applied at C″, C‴, &c., being found in a similar manner, the third condition of equilibrium of the forces which lie in the plane of x, y becomes

$$\begin{aligned} &P'(y' \cos \alpha' - x' \cos \beta') + P''(y'' \cos \alpha'' - x'' \cos \beta'') \\ &+ P'''(y''' \cos \alpha''' - x''' \cos \beta''') + \&c. = 0 \; \ldots\ldots \; (65). \end{aligned}$$

130. The six equations of equilibrium (60), (61), (62), (63), (64), (65), may be written under the following form:

$$\left.\begin{aligned} \Sigma(P \cos \alpha) &= 0 \\ \Sigma(P \cos \beta) &= 0 \\ \Sigma(P \cos \gamma) &= 0 \end{aligned}\right\} \ldots\ldots (66).$$

$$\left.\begin{aligned} \Sigma[P(y \cos \alpha - x \cos \beta)] &= 0 \\ \Sigma[P(x \cos \gamma - z \cos \alpha)] &= 0 \\ \Sigma[P(y \cos \gamma - z \cos \beta)] &= 0 \end{aligned}\right\} \ldots\ldots (67).$$

131. If there be a fixed point in the system, the six equations will not be requisite to express the conditions of equilibrium. For, if the origin be placed at the fixed point, the equilibrium will subsist between the forces acting in the plane of x, y, when the system has no tendency to turn about this point. This condition will be fulfilled when we have

$$\Sigma[P(y \cos \alpha - x \cos \beta)] = 0.$$

It remains to discover the conditions of equilibrium of the forces parallel to the axis of z. Let $x_{,,}$ $y_{,,}$ and 0 be the co-ordinates of the point at which the resultant of the parallel forces intersects the plane of x, y; the moment of this resultant taken with reference to the planes of x, z, and y, z, will be equal to the sum of the moments of the several forces taken with reference to the same planes; whence we have

$$Rx_{,}=\Sigma[P(x\cos\gamma-z\cos\alpha)],$$
$$Ry_{,}=\Sigma[P(y\cos\gamma-z\cos\beta)].$$

If an equilibrium subsists between the parallel forces, their resultant must pass through the fixed point, which, by hypothesis, coincides with the origin of co-ordinates, and we therefore have $x_{,}=0$, $y_{,}=0$. The preceding equations will thus be reduced to

$$\Sigma[P(x\cos\gamma-z\cos\alpha)]=0,$$
$$\Sigma[P(y\cos\gamma-z\cos\beta)]=0.$$

We therefore conclude that when the system contains a fixed point, the equilibrium will subsist, if the equations (67) are alone satisfied, the origin being taken at the fixed point.

132. When the system contains two fixed points, one of the co-ordinate axes may be drawn through them; this axis will thus become fixed, and the system can only be subject to a motion around it. A similar case will be examined in the succeeding paragraph.

133. When there exists a fixed axis about which the system may turn, this axis may be assumed as the axis of z, and the forces parallel to it will produce no effect. The remaining forces are situated in the plane of x, y. But the condition of equilibrium of these forces requires that their resultant should pass through the point A (*Fig.* 67), which point is immoveable, being on the axis of z; and the condition of the resultant's passing through A is expressed, as above, by the equation

$$\Sigma[P(y\cos\alpha-x\cos\beta)]=0.$$

This equation expresses that the system is in equilibrio, when the axis of z is supposed fixed.

134. If we suppose, successively, the axes of y and x to become fixed, it may in like manner be demonstrated that the system will be in equilibrio, in the first case, when

$$\Sigma[P(x\cos\gamma-z\cos\alpha)]=0,$$

and in the second, when

$$\Sigma[P(y\cos\gamma-z\cos\beta)]=0.$$

135. When the body is capable of sliding along the fixed

axis, supposed to be that of z, an additional condition of equilibrium becomes necessary; this condition is expressed by the equation

$$\Sigma(\mathrm{P} \cos \gamma)=0.$$

136. By comparing the conditions of equilibrium of a system moveable about a fixed axis, with those which obtain when the system turns about a fixed point, we infer, *That an equilibrium will take place about the fixed point when, by regarding the axes passing through this point as fixed in succession, the equilibrium is maintained with reference to each of them.*

137. If the forces be supposed to act against a fixed plane, which may be assumed as the plane of x, y, the components perpendicular to it will be destroyed by the reaction of the plane, and the conditions of equilibrium will thus be reduced to those of forces acting in a plane; we consequently have

$$\Sigma(\mathrm{P} \cos \alpha)=0,$$
$$\Sigma(\mathrm{P} \cos \beta)=0,$$
$$\Sigma[\mathrm{P}(y \cos \alpha-x \cos \beta)]=0.$$

138. If a body be supposed placed on a fixed plane, being at the same time liable to be overturned by the action of the forces exerted upon it, we must add to these three equations the condition, that the resultant of the perpendicular forces shall pass through a point in which the body touches the plane, or that it shall intersect the plane within the polygon formed by connecting the points of contact.

139. The discussion of this subject will be terminated by the solution of the following problem: *To find the analytical condition expressive of the existence of a single resultant of any number of forces situated in space.* The system will admit of a single resultant, when the resultant of the components parallel to the axis of z intersects the plane of x, y, in a point situated on the resultant of the forces lying in that plane. To express this condition, we remark, that in case of an equilibrium, the following relations must subsist between the forces parallel to the axis of z (Art. 128):

$$\mathrm{P} \cos \gamma+\mathrm{P}' \cos \gamma'+\mathrm{P}'' \cos \gamma''+\&c.=0.$$

$$P(x \cos \gamma - z \cos \alpha) + P'(x' \cos \gamma' - z' \cos \alpha')$$
$$+ P''(x'' \cos \gamma'' - z'' \cos \alpha'') + \&c. = 0.$$
$$P(y \cos \gamma - z \cos \beta) + P'(y' \cos \gamma' - z' \cos \beta')$$
$$+ P''(y'' \cos \gamma'' - z'' \cos \beta'') + \&c. = 0.$$

If we consider P cos γ, the first of these forces, as equal and directly opposed to the resultant Z of all the others, we shall have $Z = -P \cos \gamma$, and

$$-P \cos \gamma = P' \cos \gamma' + P'' \cos \gamma'' + \&c.$$
$$-P(x \cos \gamma - z \cos \alpha) = P'(x' \cos \gamma' - z' \cos \alpha')$$
$$+ P''(x'' \cos \gamma'' - z'' \cos \alpha'') + \&c.$$
$$-P(y \cos \gamma - z \cos \beta) = P'(y' \cos \gamma' - z' \cos \beta')$$
$$+ P''(x'' \cos \gamma'' - z'' \cos \beta'') + \&c.$$

The point of application of the resultant being supposed in the plane of x, y, let $x_{,}$, $y_{,}$, and 0 be the co-ordinates of this point; these values, being substituted in the first members of the preceding equations, give

$$-P \cos \gamma = P' \cos \gamma' + P'' \cos \gamma'' + \&c.,$$
$$-P \cos \gamma x_{,} = P'(x' \cos \gamma' - z' \cos \alpha')$$
$$+ P''(x'' \cos \gamma'' - z'' \cos \alpha'') + \&c.,$$
$$-P' \cos \gamma y_{,} = P'(y' \cos \gamma' - z' \cos \beta')$$
$$+ P''(y'' \cos \gamma'' - x'' \cos \beta'') + \&c.;$$

and denoting by M and N the second members of the two last equations, and replacing the factor $-P \cos \gamma$ by its value Z, we obtain

$$Z = P' \cos \gamma' + \&c.,$$
$$Zx_{,} = M,$$
$$Zy_{,} = N;$$

whence we deduce

$$x_{,} = \frac{M}{Z}, \quad y_{,} = \frac{N}{Z}.$$

Having thus obtained the values of the co-ordinates of the point at which the resultant of the parallel forces intersects the plane of x, y, it remains to express the condition that this point shall be found on the direction of the resultant of those forces which are situated in the plane of x, y; the equation of the latter resultant (Art. 111) is

$$Xy-Yx=\Sigma[P(y\cos\alpha-x\cos\beta)];$$

and putting, for brevity,

$$\Sigma[P(y\cos\alpha-x\cos\beta)]=L,$$

it becomes

$$Xy-Yx=L;$$

replacing x and y in this equation by the values of $x_{,}$ and $y_{,}$ determined above, the required condition will be expressed, and we shall obtain

$$\frac{XN}{Z}-\frac{YM}{Z}=L;$$

or, by reduction,

$$XN=LZ+MY \ldots\ldots (68).$$

If this equation be satisfied, the system will admit of a single resultant, except in the case when

$$X=0,\quad Y=0,\quad Z=0.$$

140. When the forces are situated in the same plane, the system will in general admit of a single resultant; for the quantities M and N which represent the sums of the moments taken with reference to the planes of x, z, and y, z, being equal to zero, as also the quantity Z which expresses the sum of the components $P'\cos\gamma'$, $P''\cos\gamma''$, &c., the equation (68) will be satisfied.

141. It appears from Art. 114 that the equations $X=0$ and $Y=0$ express the condition that the forces lying in the plane of x, y may be reduced to two equal resultants R' and R'', parallel to each other, and acting in contrary directions. By a similar process, the forces parallel to the axis of z may be reduced to two, Z' and Z'', equal and acting in contrary directions. Hence, when we have simply the conditions $X=0$, $Y=0$, and $Z=0$, the system may be reduced to four forces R', R'', Z', Z''. These may be still further reduced to two equal forces, having parallel and contrary directions.

Theory of the principal Plane, and Analogy existing between Projections and Moments.

142. The theory of the principal plane, which presents results so nearly allied to those obtained in the theory of mo-

ments, is of such importance in the higher branches of mechanics, as to forbid its omission in an elementary treatise. It is founded on a theorem demonstrated in the elementary treatises on the Differential Calculus, which may be enunciated as follows: *The projection of a plane surface upon a plane is equal to the area of this surface multiplied by the cosine of the angle of inclination.*

It follows, from this theorem, that if φ represent the angle formed by two planes, and λ the area of a surface situated in the first plane, the projection of this area on the second plane will be expressed by $\lambda \cos \varphi$. But the angle φ included between the two planes MF and EN (*Fig.* 68) is equal to that included between the two perpendiculars demitted from a point C on these planes. If one of these planes, EN for example, be supposed that of x, y, the perpendicular AH will become parallel to the axis of z. Thus the angle formed by the plane MF with that of x, y, is measured by the angle included between the perpendicular BK and the line AH parallel to the axis of z.

In general, if α, β, and γ represent the angles formed by the perpendicular to a given plane with the three co-ordinate axes of x, y, and z, these angles will measure the inclinations of the assumed plane to the planes of y, z, x, z, and x, y, respectively.

143. Let α, β, γ, and ι, ι', ι'', represent the angles formed respectively by any two planes with the three co-ordinate planes, these angles being equal to those formed by the perpendiculars to the given planes with the axes of co-ordinates. By introducing the cosines of these angles in the formula expressing the value of the cosine of the angle included between two lines, the value of their inclination φ may be determined.

If we draw through the point C (*Fig.* 69) the lines CA and CB perpendicular to the given planes, these lines will contain between them the angle φ, and its value will result from the formula

$$\cos \varphi = \cos \iota \cos \alpha + \cos \iota' \cos \beta + \cos \iota'' \cos \gamma \ldots\ldots (71).$$

144. When the angle φ is a right angle, its cosine will be equal to zero, and the equation becomes

$$\cos\iota\cos\alpha+\cos\iota'\cos\beta+\cos\iota''\cos\gamma=0.$$

145. From the formula (71) we deduce a very remarkable property of projections. For, let there be two planes, the first of which forms with the co-ordinate planes the angles a, b, and c, and the second the angles α, β, γ; the angle φ included between these planes being deduced from the formula (71), we have

$$\cos\varphi=\cos a\cos\alpha+\cos b\cos\beta+\cos c\cos\gamma.$$

But if we represent by λ the area of a plane surface situated in the first plane, the preceding equation being multiplied by λ, gives

$$\lambda\cos\varphi=\lambda\cos a\cos\alpha+\lambda\cos b\cos\beta+\lambda\cos c\cos\gamma\ldots\ldots(72).$$

The product $\lambda\cos\varphi$ is equal (Art. 142) to the projection of the area λ on the second plane, and the products $\lambda\cos a$, $\lambda\cos b$, and $\lambda\cos c$ are, in like manner, the projections of the same area on the co-ordinate planes.

146. The equation (72) therefore gives rise to the following theorem: *The projection of a plane surface on any plane is equal to the sum of the products of its projections on each of the co-ordinate planes, multiplied respectively by the cosines of the angles α, β, and γ, which measure the inclinations of the plane of projection to the co-ordinate planes.*

This theorem becomes much more general, if, instead of the area λ lying in a single plane, we consider several areas λ, λ', λ'', &c. situated in different planes, and projected on a plane whose inclinations to the co-ordinate planes are denoted by α, β, and γ: to avoid repetition, let us call the plane of projection α, β, γ, and denote by

$$\left.\begin{array}{r}\varphi\text{ and}\\ a,\ b,\ c,\end{array}\right\}\text{ the inclinations of the area }\lambda\left\{\begin{array}{l}\text{to the plane }\alpha,\beta,\gamma,\text{ and}\\ \text{to the co-ordinate planes,}\end{array}\right.$$

$$\left.\begin{array}{r}\varphi'\text{ and}\\ a',\ b',\ c',\end{array}\right\}\text{ the inclinations of the area }\lambda'\left\{\begin{array}{l}\text{to the plane }\alpha,\beta,\gamma,\text{ and}\\ \text{to the co-ordinate planes.}\end{array}\right.$$

$$\left.\begin{array}{r}\varphi''\text{ and}\\ a'',b'',c'',\end{array}\right\}\text{ the inclinations of the area }\lambda''\left\{\begin{array}{l}\text{to the plane }\alpha,\beta,\gamma,\text{ and}\\ \text{to the co-ordinate planes,}\end{array}\right.$$

&c. &c. &c.

By a method similar to that in which equation (72) was obtained, we can obtain similar expressions for the projections of the different areas; thus,

$$\begin{aligned}
&\lambda \cos\varphi=\lambda\cos a\cos\alpha+\lambda\cos b\cos\beta+\lambda\cos c\cos\gamma,\\
&\lambda'\cos\varphi'=\lambda'\cos a'\cos\alpha+\lambda'\cos b'\cos\beta+\lambda'\cos c'\cos\gamma,\\
&\lambda''\cos\varphi''=\lambda''\cos a''\cos\alpha+\lambda''\cos b''\cos\beta+\lambda''\cos c''\cos\gamma,\\
&\quad\&c.\qquad\qquad\&c.\qquad\qquad\&c.\qquad\qquad\&c.;
\end{aligned}$$

whence, by addition,

$$\left.\begin{aligned}
&\lambda\cos\varphi+\lambda'\cos\varphi'+\lambda''\cos\varphi''+\&c.\\
&=(\lambda\cos a+\lambda'\cos a'+\lambda''\cos a''+\&c.)\cos\alpha\\
&+(\lambda\cos b+\lambda'\cos b'+\lambda''\cos b''+\&c.)\cos\beta\\
&+(\lambda\cos c+\lambda'\cos c'+\lambda''\cos c''+\&c.)\cos\gamma
\end{aligned}\right\}\ldots\ldots(73).$$

The first member of this equation is the sum of the projections of the areas λ, λ', λ'', &c. on the plane α, β, γ; and the terms included within the brackets express the sums of the projections of the same areas on the co-ordinate planes. We therefore conclude that the enunciation of the theorem in Art. 146 will, in the present case, require to be so modified, that we may substitute in the place of the plane area λ, a surface composed of any number of plane areas λ, λ', λ'', &c. situated in different planes: this modification renders the theorem much more general.

147. For the purpose of simplifying the last equation, let us denote by P the sum of the projections of the areas λ, λ', λ'', &c. on the plane α, β, γ, and by A, B, and C the respective sums of the projections of the same areas on the three co-ordinate planes; the equation will thus be reduced to

$$P=A\cos\alpha+B\cos\beta+C\cos\gamma\ldots\ldots(74)$$

148. It should be observed, in taking the sums of these projections, that the cosines of the angles which enter into the expressions are positive or negative, according to the values of a, b, c, a', b', c', &c; thus, these sums will occasionally be changed into differences. For this reason, we should understand the enunciation of the general theorem as being applicable to the algebraic sums of the projections.

149. Let the areas λ, λ', λ'', &c. be now projected on two other planes which form with the co-ordinate planes the angles α', β', γ', α'', β'', γ''; and denote by P′ and P″ the sums of the projections of λ, λ', λ'', &c. on the planes α', β', γ', α'', β'', γ'', respectively; we shall obtain equations similar to

(74), and if we represent, as above, by A, B, and C, the sums of the projections of λ, λ', λ'', &c. on the co-ordinate planes, we shall have

$$\left.\begin{array}{l} P\ =A\cos\alpha\ +B\cos\beta\ +C\cos\gamma \\ P'=A\cos\alpha'+B\cos\beta'+C\cos\gamma' \\ P'=A\cos\alpha''+B\cos\beta''+C\cos\gamma'' \end{array}\right\}\ldots\ldots(75).$$

150. If the planes upon which the projections P, P′, and P′′ are made be supposed rectangular, their intersections will be perpendicular to each other, and may therefore be regarded as three rectangular axes, which intersect at a point O; consequently, by representing these new axes by Ox' Oy', and Oz', they will be respectively perpendicular to the new planes of co-ordinates; but the axes of x, y, and z were likewise perpendicular to the primitive co-ordinate planes; hence, the angles formed by the primitive axes with the new, will be measured by the inclinations of the primitive co-ordinate planes to the new. These angles of inclination are, by hypothesis α, β, γ; α', β', γ'; α'', β'', γ''; and since each of the primitive axes corresponds to the same letter although differently accented, we find that

The axis of x forms with the new axes the angles α, α', α'',
The axis of y forms with the new axes the angles β, β', β'',
The axis of z forms with the new axes the angles γ, γ', γ''.

The following relations will therefore subsist between the cosines of these angles,

$$\left.\begin{array}{l} \cos^2\alpha+\cos^2\alpha'+\cos^2\alpha''=1 \\ \cos^2\beta+\cos^2\beta'+\cos^2\beta''=1 \\ \cos^2\gamma+\cos^2\gamma'+\cos^2\gamma''=1 \end{array}\right\}\ldots\ldots(76).$$

Again, since the angle formed by any two of the primitive axes is a right angle, we shall obtain (Art. 144)

$$\left.\begin{array}{l} \cos\alpha\cos\beta+\cos\alpha'\cos\beta'+\cos\alpha''\cos\beta''=0 \\ \cos\alpha\cos\gamma+\cos\alpha'\cos\gamma'+\cos\alpha''\cos\gamma''=0 \\ \cos\beta\cos\gamma+\cos\beta'\cos\gamma'+\cos\beta''\cos\gamma''=0 \end{array}\right\}\ldots\ldots(77).$$

151. If we take the sum of the squares of the equations (75), reducing by means of (76) and (77), we shall obtain the relation

$$P^2+P'^2+P''^2=A^2+B^2+C^2\ \ldots\ldots(78);$$

which expresses that the sum of the squares of the projections of the areas λ, λ', λ'', &c. on any three rectangular planes is a constant quantity.

152. Several important consequences may be deduced from this theorem: thus, if we resolve the equation (78) with reference to P, we find

$$P=\sqrt{(A^2+B^2+C^2-P'^2-P''^2)}.$$

The value of P will evidently be greatest when P′ and P″ are equal to zero. In this case, the sum of the projections of λ, λ', λ'', &c. on the plane α, β, γ, will be given by the equation

$$P=\sqrt{(A^2+B^2+C^2)} \;.\;.\;.\;.\;.\; (79).$$

But the angles α, α', α'', being the angles formed by the primitive axis of x, with the three new axes, we must have the relation

$$A=P\cos\alpha+P'\cos\alpha'+P''\cos\alpha'';$$

and by considering the other angles, we obtain in like manner,

$$B=P\cos\beta+P'\cos\beta'+P''\cos\beta'',$$
$$C=P\cos\gamma+P'\cos\gamma'+P''\cos\gamma''.$$

If we suppose, as above, the quantities P′ and P″ to be equal to zero, the preceding equations reduce to

$$A=P\cos\alpha,\quad B=P\cos\beta,\quad C=P\cos\gamma \;.\;.\;.\;.\;.\; (80).$$

whence,

$$\cos\alpha=\frac{A}{P},\quad \cos\beta=\frac{B}{P},\quad \cos\gamma=\frac{C}{P},$$

and by substituting for P its value given in equation (79), we find

$$\left.\begin{aligned}\cos\alpha&=\frac{A}{\sqrt{(A^2+B^2+C^2)}}\\ \cos\beta&=\frac{B}{\sqrt{(A^2+B^2+C^2)}}\\ \cos\gamma&=\frac{C}{\sqrt{(A^2+B^2+C^2)}}\end{aligned}\right\} \;.\;.\;.\;.\;.\; (81).$$

These angles express the inclinations of the plane of maximum projections, which is called the principal plane.

The determination of this plane being dependent only on the angles α, β, γ, the same property will be enjoyed by every parallel plane.

153. It may also be demonstrated that the sum of the projections of the areas λ, λ', λ'', &c., on every plane which is equally inclined to the principal plane, will be equal to a constant quantity. For, let Q be the sum of the projections on any plane whose inclinations to the co-ordinate planes are denoted by a, b, and c: if we represent, as heretofore, by A, B, C the projections of these areas on the co-ordinate planes, we shall have

$$Q=A\cos a+B\cos b+C\cos c;$$

but if α, β, γ denote the inclinations of the principal plane, the equations (80), which are

$$A=P\cos\alpha,\quad B=P\cos\beta,\quad C=P\cos\gamma,$$

will reduce the preceding equation to

$$Q=P(\cos a\cos\alpha+\cos b\cos\beta+\cos c\cos\gamma).$$

The quantity within the brackets being equal to the cosine of the angle included between the principal plane α, β, γ and the assumed plane a, b, c, we shall have, by calling this inclination θ,

$$Q=P\cos\theta;$$

and since P represents the sum of the projections on the principal plane, which, by Art. 152, is equal to $\sqrt{(A^2+B^2+C^2)}$, the substitution of this value gives

$$Q=\sqrt{(A^2+B^2+C^2)}\times\cos\theta\,.....(82).$$

But the projections A, B, and C remaining the same, it follows from the equation (82) that the value of Q, the sum of the projections on any plane, will be constantly the same for all planes having the same inclination to the principal plane.

It also appears that this sum will increase or diminish in the same ratio as $\cos\theta$.

154. Lastly, it may be remarked that the sum of the projections on every plane perpendicular to the principal plane is equal to zero; for $\theta=90^\circ$ gives $\cos\theta=0$, and $Q=0$.

155. The several theorems relative to projections which have just been demonstrated are likewise applicable to the case of moments. For, let the centre of moments be supposed to coincide with the origin of co-ordinates, and conceive the plane α, β, γ to pass through the origin: if from the

points of application of the several forces we take upon their respective lines of direction, portions which shall be proportional to the intensities of these forces, these lines may be represented by the letters P, P′, P″, &c. The centre of moments may then be regarded as the common vertex of several triangles, of which P, P′, P″, &c. represent the bases: the projections of these triangles upon the plane α, β, γ, and on the co-ordinate planes will likewise be triangles, their bases p, p', p'', &c., being the projections of the lines P, P′, P″, &c., and their altitudes h, h', h'', &c., being the perpendiculars demitted on the lines p, p', p'', &c. from the centre of moments.

These values being substituted in equation (73), which may be written under the following form:

$$\Sigma(\text{the projections on the plane } \alpha, \beta, \gamma) = \Sigma\left\{\begin{array}{l}\text{The projections on the co-ordinate planes multiplied}\\ \text{respectively by the cosines of the angles of inclination}\end{array}\right\},$$

convert it into

$$\tfrac{1}{2}ph + \tfrac{1}{2}p'h' + \tfrac{1}{2}p''h'' + \&\text{c.} = \Sigma\left\{\begin{array}{l}\text{The projections on the co-ordinate}\\ \text{planes multiplied respectively by the}\\ \text{cosines of the angles of inclination}\end{array}\right\} \ldots\ldots (83).$$

The second member of this equation will contain similar products, and the factor $\frac{1}{2}$ will therefore be common to the two members; this being suppressed, the first member will reduce to

$$ph + p'h' + p''h'' + \&\text{c.}$$

But p, p', p'', &c., being the projections of the right lines P, P′, P″, &c., the products ph, $p'h'$, $p''h''$, &c. will be the moments of the lines p, p', p'', &c., taken with reference to the origin of co-ordinates. The same remarks being applicable to the second member of equation (83), it follows that the sum of the moments of the projections of the forces on the plane α, β, γ, which passes through the origin of co-ordinates, is equal to the sum of the moments of the projections of the same forces on the three co-ordinate planes, multiplied respectively by the cosines of the angles of inclination.

156. By making similar substitutions in equations (78), it may likewise be proved that the sum of the squares of the

moments of the different forces, when projected on three rectangular planes, is a constant quantity.

The equations (80) make known the position of the plane in which the sum of the moments will be the greatest possible. And the equation (79) determines the sum of the moments on the principal plane.

Centre of Gravity.

157. The particles of matter are constantly subjected to the action of a force which tends to draw them towards the earth, in directions perpendicular to its surface. This force is called the force of gravity.

The earth being nearly spherical, the lines of direction in which material points tend to move, will converge towards its centre; and since the distance of this centre from the surface is exceedingly great when compared with the dimensions of those objects which we usually consider, the directions of the forces which act on the different particles of the same body may, without sensible error, be regarded as parallel.

158. It is known from observation that, as we recede from the centre of the earth, the intensity of gravity diminishes in the inverse ratio of the square of the distance included between the centre and the place of observation. For example, if a body be placed at a certain distance from the centre of the earth, assumed as unity, and be subsequently transported to distances represented by 2, 3, 4, &c., the intensity of the force of gravity will become $\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}$, &c., or $\frac{1}{4}, \frac{1}{9}, \frac{1}{16}$, &c., of what it was at the distance of unity.

159. The earth being flattened towards the poles, and protuberant at the equator, it follows, that in going from the equator towards the poles, we must necessarily approach the centre of the earth, and the intensity of gravity will therefore increase. It will appear hereafter in discussing the subject of centrifugal forces, that from another cause, the intensity of the force of gravity is greater at the poles than at all other places on the earth's surface.

160. The action of gravity being exerted on all the particles which compose a body, these particles may be regarded as solicited by forces whose directions are parallel; the resultant of these forces is equal to their sum, and constitutes what is called the *weight* of a body. Hence, if the bodies considered are homogeneous with each other, their weights will be proportional to their volumes.

161. The term density is used to express the greater or less number of particles contained in a body of a given volume, when compared with the number of particles contained in some other body assumed as a standard. If we assume as the unit, the quantity of matter contained in a cubic foot of a given substance, distilled water for example, and compare this quantity with that contained in a cubic foot of any other substance, their ratio will express the *density* of the second substance. Let this ratio be denoted by D. If the second substance considered were gold, by calling D the density of gold, we should have

$$\left.\begin{matrix}\textit{The quantity of matter in}\\ \textit{a cubic foot of gold}\end{matrix}\right\} = \left\{\begin{matrix}\mathrm{D}\times\textit{The quantity of matter}\\ \textit{in a cubic foot of water};\end{matrix}\right.$$

whence

$$\mathrm{D}=\frac{\textit{Quantity of matter in a cubic foot of gold}}{\textit{Quantity of matter in a cubic foot of water}}.$$

162. In the preceding article we have considered bodies of the same volume; but if we wish to estimate the quantity of matter contained in a homogeneous body whose volume is V, the quantity D must be taken as many times as there are units of volume in the volume V; we shall thus have

$$\mathrm{M}=\mathrm{DV} \ldots\ldots (84).$$

The quantity M is called the *mass*, and evidently expresses the relation between the quantity of matter contained in the body, and that contained in the unit of volume of the substance assumed as the standard.

163. If the intensity of gravity were the same at all places, the weight of a body would be proportional to its mass, and might be represented by the same quantity. For, if g denote the effect exerted by gravity on the unit of mass, or the weight of the unit of mass, and W the weight of the body, we

shall have, from the definition of the weight, $W = Mg$; in which expression the quantity g will be constant, and may be assumed as the unit; we shall thus obtain the relation

$$W = M \ldots\ldots (85).$$

This equation merely expresses that the number of units of weight is equal to the number of units of mass.

But, if by transporting the mass to different distances from the earth's centre, the intensity of gravity be subject to variation, the quantity g will be variable, and the equation expressing the relation between the mass, weight, and intensity of gravity, must then be written under the general form

$$W = Mg \ldots\ldots (86).$$

164. From the equations (84) and (86), we deduce

$$W = DVg;$$

which indicates that *the weight varies proportionally to the gravity g, the volume V, and the density D.*

165. If, for example, two bodies of the same volume be subjected to the action of the same force of gravity, their weights will be in the direct ratio of their densities.

The intensity of gravity varying only with change of place, it follows that g will be constant for all bodies at the same place.

166. If there be any number of points firmly connected together, and solicited by the weights P, P', P'', &c., we may regard these weights as parallel forces; and denoting the co-ordinates of the respective points by x, y, z, x', y', z', x'', y'', z'', &c., we shall obtain, from Art. (80) and (81), the expressions for the co-ordinates of the centre of parallel forces; these co-ordinates being represented by $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, we find

$$x_{\prime} = \frac{Px + P'x' + P''x'' + \&c.}{P + P' + P'' + \&c.},$$

$$y_{\prime} = \frac{Py + P'y' + P''y'' + \&c.}{P + P' + P'' + \&c.},$$

$$z_{\prime} = \frac{Pz + P'z' + P''z'' + \&c.}{P + P' + P'' + \&c.}.$$

167. When the forces are exerted, as in the present instance, by the action of gravity, the centre of parallel forces is called the *centre of gravity.* Let m, m', m'', &c. represent

the masses corresponding to the weights P, P', P'', &c., we shall have

$$P=mg, \quad P'=m'g, \quad P''=m''g, \text{ \&c.};$$

and by substituting these values in the preceding equations, omitting the factor g, which is common to the numerators and denominators of the fractions, we obtain

$$x_{\prime}=\frac{mx+m'x'+m''x''+\text{\&c.}}{m+m'+m''+\text{\&c.}},$$

$$y_{\prime}=\frac{my+m'y'+m''y''+\text{\&c.}}{m+m'+m''+\text{\&c.}},$$

$$z_{\prime}=\frac{mz+m'z'+m''z''+\text{\&c.}}{m+m'+m''+\text{\&c.}};$$

whence it appears that the position of the centre of gravity is independent of the intensity of the force of gravity.

168. If the bodies are composed of a homogeneous substance, the density of which is represented by D, we shall have, by denoting their volumes by v, v', v'', &c. (Art. 162),

$$m=v\mathrm{D}, \quad m'=v'\mathrm{D}, \quad m''=v''\mathrm{D}, \text{ \&c.};$$

and by a substitution and reduction similar to the preceding, we find

$$x_{\prime}=\frac{vx+v'x'+v''x''+\text{\&c.}}{v+v'+v''+\text{\&c.}},$$

$$y_{\prime}=\frac{vy+v'y'+v''y''+\text{\&c.}}{v+v'+v''+\text{\&c.}},$$

$$z_{\prime}=\frac{vz+v'z'+v''z''+\text{\&c.}}{v+v'+v''+\text{\&c.}};$$

or calling V the volume of the entire system, these equations become

$$x_{\prime}=\frac{vx+v'x'+v''x''+\text{\&c.}}{\mathrm{V}},$$

$$y_{\prime}=\frac{vy+v'y'+v''y''+\text{\&c.}}{\mathrm{V}},$$

$$z_{\prime}=\frac{vz+v'z'+v''z''+\text{\&c.}}{\mathrm{V}}$$

169. To determine the centre of gravity experimentally, we suspend the body by a thread CA (*Fig.* 70), and the prolongation AB of the direction of this thread will necessarily

pass through the centre of gravity. The point in the line AB at which the centre of gravity is situated, may then be found by suspending the body from a second point E; the vertical line EF, passing through this point, must likewise pass through the centre of gravity, which will consequently be found at the point G, the intersection of the two lines AB and EF.

In this experiment, the body is sustained by that point to which the thread is attached: the resultant of all the actions of gravity upon the particles of the body must therefore pass through this point, and its direction must coincide with that of the thread.

170. *The centre of gravity of a right line* AB (*Fig.* 71) *is situated at its middle point* C: for, by regarding the line as composed of heavy material points, each particle m situated on one side of the point C will correspond to a particle m' on the contrary side, and equally distant from the same point: the moments $m \times Cm$ and $m' \times Cm'$ are therefore equal and have contrary signs. The same remarks are applicable to all the other points of the line AB, taken by pairs; hence it follows, that the algebraic sum of the moments of all the particles taken with reference to the point C is equal to zero; the moment of the resultant taken with reference to the same point is therefore zero, and the direction of the resultant must pass through the point C, situated in the middle of the line AB.

171. *The centre of gravity of a parallelogram* AD (*Fig.* 72) *is at the intersection* G *of the right lines* EF *and* HK, *which bisect the parallel sides.*

For, if we conceive the particles which compose the parallelogram to be situated on lines parallel to AB, the centres of gravity of all these lines will be found on the line EF drawn through the middle points E and F of the opposite sides AB and CD, since EF will bisect all these parallels. Hence, the centre of gravity of the entire parallelogram will be situated on the line EF. In like manner, it may be proved that the centre of gravity lies on the line HK which bisects the sides AC and BD; it will therefore be situated at the point G, the intersection of the two lines EF and HK.

172. *The centre of gravity* G *of the area of a triangle* ABC (*Fig.* 73) *is found by drawing a line* CD *from the vertex to the middle of the opposite side, and taking a part* DG *equal to one-third of the whole line* CD. For, since the line CD passes through the middle of all the lines parallel to the base AB, it contains the centre of gravity of the area of the triangle: for a similar reason, this centre must lie on the line AE drawn through the middle of the side CB: hence, it is found at the point G, the intersection of these two lines. But, by connecting the points D and E, we form the triangle BED, which is similar to the triangle BCA, since the two triangles have a common angle, and the sides adjacent directly proportional: the line DE is therefore parallel to AC, and the triangles ACG and DEG are likewise similar; hence,

$$CG : GD :: AC : DE :: AB : BD :: 2 : 1;$$

from which results

$$CG = 2GD,$$

and, consequently,

$$CD \text{ or } CG + GD = 3GD,$$

or,

$$GD = \tfrac{1}{3}CD.$$

173. *To find the centre of gravity of a triangular pyramid, we draw through the vertex and the centre of gravity of the base, the line* AG (*Fig.* 74), *and take the distance* $GO = \frac{1}{4}AG$; *the point* O *will be the centre of gravity of the pyramid.*

For, if we conceive the pyramid divided into an infinite number of elements by planes parallel to the base BCD; the line AG will pass through the centres of gravity of all these elements, and will therefore contain the centre of gravity of the pyramid. In like manner, by drawing the line DG′ through the vertex D and the centre of gravity G′ of the opposite face, this line will also contain the centre of gravity of the pyramid. But, since the lines AG and DG′ are situated in the plane of the triangle AED, and are not parallel, they will intersect, and hence the centre of gravity of the pyramid will be found at O, their point of intersection.

The points G and G′ being connected, the triangles EGG′

and EDA will be similar, since they have a common angle E, and the sides containing it directly proportional; hence, GG′ is parallel to AD, and the triangles AOD, GOG′ are likewise similar; from these we obtain

$$GG' : AD :: GO : OA;$$

but the similar triangles EGG′ and EDA give

$$GG' : AD :: EG : ED;$$

whence, by comparing these two proportions, we have

$$GO : OA :: EG : ED :: 1 : 3;$$

and from this proportion we find

$$3GO = OA;$$

adding GO to each member of the equation, there results

$$4GO = OA + GO = AG,$$

or,

$$GO = \tfrac{1}{4}AG.$$

174. In general, the centre of gravity of any pyramid (*Fig.* 75) is situated on the right line SF, drawn from the vertex to the centre of gravity of the base, and at a distance $FO = \frac{1}{4}SF$. If we draw through the point O thus determined, a plane parallel to the base of the pyramid, this plane will contain the centre of gravity of the pyramid. For, if through F, the centre of gravity of the polygonal base, the lines FA, FB, &c. be drawn to the several angles of this polygon, we shall form as many triangles as the figure has sides, and these triangles may be regarded as the bases of triangular pyramids having a common vertex S. The lines drawn from the vertex S to the centres of gravity of the several triangles will be cut proportionally by the plane parallel to the base, and the points of intersection will therefore be situated at distances from the base, equal to one-fourth of the distance from the base to the vertex of the pyramid. Hence, these points of intersection will be the centres of gravity of the several triangular pyramids. But the centres of gravity of all the partial pyramids being situated in the same plane parallel to the base, it follows that the centre of gravity of the whole pyramid will likewise be situated in this plane. It must also be found on the line SF, which contains the centres

of gravity of all the sections parallel to the base, and we therefore conclude that *the centre of gravity of any pyramid is situated on the line drawn from the vertex of the pyramid to the centre of gravity of the base, and at a distance from the base equal to one-fourth of the entire distance to the vertex.*

175. *To find the centre of gravity of the area of a polygon.* Let the polygon be divided into triangles (*Fig.* 76), and denote by a, a', a'', &c., the areas ABC, ACD, ADE, &c. of these triangles: let weights proportional to a, a', a'', &c. be supposed applied at the centres of gravity G, G′, G″, &c., of the several triangles. The centre of gravity of the area ABCDA may then be found by the proportion

$$a+a' : a :: \mathrm{GG'} : \mathrm{G'O}.$$

In like manner, the centre of gravity K of the area ABCDEA may be found by determining the resultant of $a+a''$ acting at O, and a'' acting at G″. Its position will be ascertained by the proportion

$$a+a'+a'' : a'' :: \mathrm{OG''} : \mathrm{OK};$$

and the same process may be continued for any number of triangles.

176. This problem may also be solved by means of the equations of parallel forces. For let $x_{,}$ and $y_{,}$ denote the coordinates of the centre of gravity of the polygon (*Fig.* 77): from the theory of parallel forces we obtain the equations

$$\mathrm{R}=\mathrm{P}+\mathrm{P'}+\mathrm{P''}+\mathrm{P'''},$$
$$\mathrm{R}x_{,}=\mathrm{P}x+\mathrm{P'}x'+\mathrm{P''}x''+\mathrm{P'''}x''',$$
$$\mathrm{R}y_{,}=\mathrm{P}y+\mathrm{P'}y'+\mathrm{P''}y''+\mathrm{P'''}y'''.$$

And denoting as above by a, a', a'', a''', the areas of the triangles ABC, ACD, ADE, AEF, we shall have, since the areas may be substituted for the weights to which they are proportional,

$$\mathrm{P}=a, \quad \mathrm{P'}=a', \quad \mathrm{P''}=a'', \quad \mathrm{P'''}=a''';$$

and the preceding equations become

$$\mathrm{R}=a+a'+a''+a''',$$
$$x_{,}=\frac{ax+a'x'+a''x''+a'''x'''}{a+a'+a''+a'''},$$
$$y_{,}=\frac{ay+a'y'+a''y''+a'''y'''}{a+a'+a''+a'''}.$$

Thus, having taken the part OP$=x_{,}$ we draw the line PG parallel to the axis of y and equal to $y_{,}$; the point G will be the centre of gravity.

177. *To find the centre of gravity of the perimeter of a polygon.* We proceed in the present case in a manner similar to that adopted in the preceding example, merely observing that the centre of gravity of each side will be situated at its middle point, and that these points may be regarded as loaded with weights proportional to the sides.

178. *To find the centre of gravity of the arc of a plane curve.* If the curve be divided into elementary portions, the value of the element mm' (*Fig.* 78) will be expressed by $\surd(dx^2+dy^2)$, and since this element is indefinitely small, its centre of gravity may be regarded as coinciding with its middle point o, and having the same co-ordinates x and y as the point m; the moment of mm' with reference to the axis of x, will therefore be

$$op\times mm'=y\times\surd(dx^2+dy^2),$$

and its moment with reference to the axis of y, will be

$$oq\times mm'=x\times\surd(dx^2+dy^2).$$

If $x_{,}$ and $y_{,}$ represent the co-ordinates of the centre of gravity, and s the length of the curve MM', the moments of this arc supposed concentrated at its centre of gravity, taken with reference to the axes, will be respectively $sx_{,}$ and $sy_{,}$: and since these moments must be equal to the sum of the moments of the elements, we shall have

$$sx_{,}=\int x\surd(dx^2+dy^2),$$
$$sy_{,}=\int y\surd(dx^2+dy^2);$$

and the length of the arc MM' will result from the formula

$$s=\int\surd(dx^2+dy^2).$$

179. Let it be required, for example, *to determine the centre of gravity of the arc* BO *of a circle* (*Fig.* 79). The co-ordinate axes being selected in such a manner that the arc shall be bisected by the axis of abscisses passing through the centre of the circle, the arc will be divided symmetrically by this axis, and the centre of gravity of the arc will then be

found on this line; hence, we shall have $y_{,}=0$. It will therefore be only necessary to determine the absciss $AG=x_{,}$ of the centre of gravity of the arc BO. But the value of $x_{,}$ results from Art. 178; thus,

$$sx_{,}=\int x\surd(dx^2+dy^2)\ .\ .\ .\ .\ .\ (87).$$

To integrate the second member of this equation, we eliminate one of the variables by means of the equation of the circle, which is

$$y^2=a^2-x^2\ .\ .\ .\ .\ .\ (88);$$

and by differentiating this equation, we obtain

$$ydy=-xdx;$$

whence,

$$dx^2=\frac{y^2dy^2}{x^2};$$

and by substituting this value in the expression $\surd(dx^2+dy^2)$, we have

$$\surd(dx^2+dy^2)=\surd\left(\frac{x^2+y^2}{x^2}dy^2\right);$$

which, reduced by means of equation (88), gives

$$\surd(dx^2+dy^2)=\frac{ady}{x};$$

this value being substituted in equation (87), we find, by integration

$$\int x\surd(dx^2+dy^2)=ay+B\ .\ .\ .\ .\ .\ (89),$$

the quantity B representing an arbitrary constant.

If we denote by c the chord of the arc BO, and wish to determine the centre of gravity of the arc which it subtends, we must integrate between the limits $y=\frac{1}{2}c$, $y=-\frac{1}{2}c$. But since the arc extends from O to B, this integral will become zero at the point O, the ordinate of which is $y=-\frac{1}{2}c$. This supposition reduces equation (89) to

$$0=-\tfrac{1}{2}ac+B;$$

by eliminating B between this equation and (89), we find

$$\int x\surd(dx^2+dy^2)=ay+\tfrac{1}{2}ac;$$

and making $y=\frac{1}{2}c$, for the purpose of taking the entire integral from the point O to the point B, we obtain

$$\int x\sqrt{(dx^2+dy^2)}=ac;$$

which value substituted in equation (87), gives

$$sx_{,}=ac,$$

or

$$x_{,}=\frac{radius\times chord}{arc}\;\ldots\ldots\;(90);$$

the abscissa of the centre of gravity is therefore a fourth proportional to the arc, the chord, and the radius.

180. *To find the centre of gravity of a curve of double curvature, or, in general, that of any line situated in space.*

The expression for the element of a curve of double curvature being

$$\sqrt{(dx^2+dy^2+dz^2)}\;\ldots\ldots\;(91),$$

let the moments of this element be taken with reference to the co-ordinate planes. The co-ordinates x, y, and z represent the distances of this element from the planes of y, z, x, z, and xy, and the respective moments will therefore be

$$\left.\begin{array}{l} x\sqrt{(dx^2+dy^2+dz^2)} \\ y\sqrt{(dx^2+dy^2+dz^2)} \\ z\sqrt{(dx^2+dy^2+dz_2)} \end{array}\right\}\;\ldots\ldots\;(92);$$

consequently, if we denote by $x_{,}$, $y_{,}$, and $z_{,}$ the co-ordinates of the centre of gravity, and by s the length of the arc, these quantities will be determined by means of the equations

$$\left.\begin{array}{l} s=\int\sqrt{(dx^2+dy^2+dz^2)} \\ sx_{,}=\int x\sqrt{(dx^2+dy^2+dz^2)} \\ sy_{,}=\int y\sqrt{(dx^2+dy^2+dz^2)} \\ sz_{,}=\int z\sqrt{(dx^2+dy^2+dz^2)} \end{array}\right\}\;\ldots\ldots\;(93).$$

181. Let it be required to apply these formulas to the case of a right line situated in space. Assume the origin at one extremity of the line; the equations of the line will then be of the form

$$x=\alpha z,\quad y=\beta z\;\ldots\ldots\;(94);$$

whence,

$$dx=\alpha dz,\quad dy=\beta dz.$$

These values substituted in the expression (91), give

$$\sqrt{(dx^2+dy^2+dz^2)}=dz\sqrt{(1+\alpha^2+\beta^2)};$$

and putting, for brevity, the radical equal to A, we shall have

$$\sqrt{(dx^2+dy^2+dz^2)}=\mathrm{A}dz.$$

Substituting this value in the equations (93), and likewise those of x and y given by equations (94), we find

$$s=\int \mathrm{A}dz=\mathrm{A}z,$$

$$sx_{\prime}=\int \mathrm{A}\alpha zdz=\tfrac{1}{2}\mathrm{A}\alpha z^2,$$

$$sy_{\prime}=\int \mathrm{A}\beta zdz=\tfrac{1}{2}\mathrm{A}\beta z^2,$$

$$sz_{\prime}=\int \mathrm{A}zdz=\tfrac{1}{2}\mathrm{A}z^2.$$

Let h represent the ordinate z of the point M (*Fig.* 80). To determine the centre of gravity of AM, we must integrate between the limits $z=0$ and $z=h$, and we shall thus find

$$s=\mathrm{A}h,$$

$$sx_{\prime}=\tfrac{1}{2}\mathrm{A}\alpha h^2,$$

$$sy_{\prime}=\tfrac{1}{2}\mathrm{A}\beta h^2,$$

$$sz_{\prime}=\tfrac{1}{2}\mathrm{A}h^2.$$

Eliminating s, and reducing, we obtain

$$x_{\prime}=\tfrac{1}{2}\alpha h,\quad y_{\prime}=\tfrac{1}{2}\beta h,\quad z_{\prime}=\tfrac{1}{2}h.$$

These values correspond to the co-ordinates of the point O, the middle of the right line AM; for, if AO be the half of AM, the similar triangles AOQ, AMP will give

$$\mathrm{QO}=\tfrac{1}{2}\mathrm{MP}=\tfrac{1}{2}h;$$

which value being substituted in equations (94), we find

$$x=\tfrac{1}{2}\alpha h,\quad y=\tfrac{1}{2}\beta h.$$

182. *To find the centre of gravity of a plane surface, bounded by the arc of a curve, and the axis of abscisses.*

Let $x_{\prime}$ and $y_{\prime}$ be the co-ordinates of the centre of gravity of the entire surface, and let G be the centre of gravity of an element MP′ (*Fig.* 81); the area of this element being equal to ydx, its moment with reference to the axis of x will be $\mathrm{GN}\times ydx$, and that with respect to the axis of y will be $\mathrm{AN}\times ydx$. But since the element MP′ may be regarded as a rectangle whose side PP′ is indefinitely small, we shall have $\mathrm{AP}=\mathrm{AN}=x$, and $\mathrm{GN}=\frac{\mathrm{PM}}{2}=\frac{1}{2}y$: hence the moments with reference to the two axes become $\frac{1}{2}y^2dx$, and $xydx$. If we represent by λ the surface DBMP, its area and the co-ordinates of its centre of gravity will be determined by means of the equations

$$\left.\begin{aligned} \lambda&=\int y\,dx, \\ \lambda x_{\prime}&=\int xy\,dx, \\ \lambda y_{\prime}&=\int \tfrac{1}{2}y^2\,dx, \end{aligned}\right\}\ldots\ldots(95),$$

183. To apply these formulas, let it be required to find the centre of gravity of a circular segment CDE (*Fig*. 82). The origin being assumed at the centre of the circle, and the axis of abscisses AD a line bisecting the arc CE, the centre of gravity of the segment will evidently be situated upon this line; it will therefore be only necessary to calculate the value of the absciss $AG=x_{\prime}$. If g and g' represent the centres of gravity of the semi-segments, they will be found at equal distances from the axis AD, on a line gg' perpendicular to this axis, since the entire segment is divided into two symmetrical portions; the line gg' will therefore intersect the axis of abscisses at a point G, the centre of gravity of the entire segment.

The question is thus reduced to determining the absciss of the centre of gravity of the semi-segment CDB, and its value may be found by integrating the equation (95).

For the purpose of eliminating one of the variables in this expression, we assume the differential equation of the circle,

$$y\,dy=-x\,dx;$$

from which, by substitution in equation (95), we obtain

$$\lambda x_{\prime}=\int -y^2\,dy \ldots\ldots (96);$$

and by integrating, and introducing a constant A, we have

$$\int -y^2\,dy=-\tfrac{1}{3}y^3+\mathrm{A} \ldots\ldots (97).$$

To determine the value of this constant, the integral must be taken from the point C to the point D; or, if we denote by c the value of the chord CE, the limits of the integral will be $y=\frac{1}{2}c$ and $y=0$. Thus, if we suppose the integral to become zero, when $y=\frac{1}{2}c$, the constant A will result from the equation

$$0=-\frac{c^3}{24}+\mathrm{A},$$

and the equation (97) will therefore become

$$\int -y^2 dy = -\tfrac{1}{3}y^3 + \frac{c^3}{24}.$$

Putting $y=0$, to obtain the value of the entire integral from C to D, we have

$$\int -y^2 dy = \frac{c^3}{24}.$$

This value substituted in equation (96), gives

$$x_i = \frac{c^3}{24\lambda}:$$

but since λ represents in this expression the area CDB, we have

$$\lambda = \tfrac{1}{2} \text{ area CDEB},$$

whence,

$$x_i = \frac{c^3}{12 \text{ area CDEB}};$$

and we therefore conclude, that *the distance from the centre of gravity of a circular segment to the centre of the circle is equal to the cube of the chord divided by twelve times the area of the segment.*

184. *To find the centre of gravity of a circular sector* CAE (*Fig.* 83). The centre of gravity is evidently situated on the radius AB which divides the sector into two equal parts; it will therefore be only necessary to determine the value of the absciss AG. If we regard the sector CAE as composed of an infinite number of elementary sectors, the centre of gravity of each will be situated at a distance from the point A equal to two-thirds of the radius AC, since these sectors may be considered triangular. Hence, if from the centre A, with a radius equal to two-thirds of AC, we describe the arc HK, the centres of gravity of all the elementary sectors will be distributed uniformly along this arc; and consequently, the centre of gravity of this arc will coincide with that of the circular sector. But if x_i denote the absciss AG, we have, by Art. 179,

$$x_i = \frac{\text{AH} \times \text{chord HK}}{\text{arc HK}};$$

and from the similarity of the sectors AHK and ACE, we find

$$\mathrm{AH}=\tfrac{2}{3}\mathrm{AC},$$

$$\text{chord HK}=\tfrac{2}{3}\text{ chord CE},$$

$$\text{arc HK}=\tfrac{2}{3}\text{ arc CE};$$

which values substituted in the preceding equation give by reduction,

$$x_{,}=\frac{\tfrac{2}{3}\mathrm{AC}\times\text{chord CE}}{\text{arc CE}}.$$

185. *To find the centre of gravity of an area* OBO′ (*Fig.* 84) *comprised between two branches of a curve.*

Let y and y' represent the two ordinates PM and PM′ corresponding to the same absciss AP$=x$: the element MN′ of the surface, being the difference of the areas PN and PN′, will be expressed by

$$ydx-y'dx=(y-y')dx;$$

and if we represent by λ a portion of the area included between the chords MM′ and OO′, we shall have

$$\lambda=\int(y-y')dx.$$

The element MN′ being regarded as a rectangle having one of its sides indefinitely small, its centre of gravity will be situated in the middle of the line MM′; and the ordinate of this point will therefore be

$$\mathrm{PM'}+\tfrac{1}{2}\mathrm{MM'}=y'+\tfrac{1}{2}(y-y')=\tfrac{1}{2}(y+y');$$

hence, the moment of this element with reference to the axis of x will be

$$\tfrac{1}{2}(y+y')(y-y')dx=\tfrac{1}{2}(y^2-y'^2)dx;$$

and the moment with reference to the axis of y will be

$$x(y-y')dx.$$

Thus, if $x_{,}$ and $y_{,}$ denote the co-ordinates of the centre of gravity of the entire surface, their values will become known from the equations

$$\lambda x_{,}=\int x(y-y')dx,$$

$$\lambda y_{,}=\int\tfrac{1}{2}(y^2-y'^2)dx.$$

186. *To find the centre of gravity of a surface of revolution.*

Let the surface be supposed generated by the revolution of the curve AM (*Fig.* 85) about the axis of x. The element of the surface, or the zone generated by the elementary arc Mm, will be expressed by $2\pi y ds$: hence, by calling λ the entire surface, we shall obtain

$$\lambda = \int 2\pi y ds.$$

But since the centre of gravity is evidently situated on the axis of revolution, the co-ordinate $x_{\prime}$ will be alone necessary. To determine its value, we take the sum of the moments with reference to the plane yz, which sum being equal to the moment of the whole surface supposed concentrated at its centre of gravity, we find

$$\lambda x_{\prime} = \int x \times 2\pi y ds;$$

whence,

$$x_{\prime} = \frac{\int 2\pi y x ds}{\lambda};$$

substituting for λ and ds their respective values, and suppressing the factor 2π common to both terms of the fraction, we obtain for the absciss of the centre of gravity,

$$x_{\prime} = \frac{\int xy\sqrt{(dx^2+dy^2)}}{\int y\sqrt{(dx^2+dy^2)}} \ldots\ldots (98).$$

187. For the purpose of applying this formula, let it be required to determine the centre of gravity of the surface of a *spheric segment*. This surface being generated by the revolution of a circular arc BC (*Fig.* 86) about the axis of x, we may eliminate one of the variables in the preceding formula by means of the equation of the circle;

$$y^2 = r^2 - x^2;$$

which gives, by differentiation,

$$dy^2 = \frac{x^2 dx^2}{y^2};$$

hence,

$$\sqrt{(dx^2+dy^2)} = \frac{dx\sqrt{(x^2+y^2)}}{y} = \frac{rdx}{y}.$$

This value being substituted in the integrals of equation (98), we find

$$\int xy\sqrt{(dx^2+dy^2)} = \int rxdx = \tfrac{1}{2} rx^2 + \mathrm{C},$$
$$\int y\sqrt{(dx^2+dy^2)} = \int rdx = rx + \mathrm{C}'.$$

Taking the integrals between the limits $x = \mathrm{AD} = a$, and $x = \mathrm{AB} = r$, we obtain

$$\int xy\sqrt{(dx^2+dy^2)}=\tfrac{1}{2}r(r^2-a^2),$$

$$\int y\sqrt{(dx^2+dy^2)}=r(r-a).$$

These values transform the equation (98) into

$$x_{\prime}=\tfrac{1}{2}(r+a)=a+\tfrac{1}{2}(r-a);$$

thus, the centre of gravity is situated at the middle of the line DB.

188. *To find the centre of gravity of a solid of revolution M, bounded by two planes perpendicular to the axis,* (*Fig.*87).

The centre of gravity being necessarily situated upon the axis of revolution, which is supposed to coincide with the axis of x, it will be sufficient to determine its absciss $x_{\prime}$. The element of the solid is expressed by $\pi y^2 dx$, and we therefore have

$$M=\int \pi y^2 dx \ldots\ldots (99).$$

The moments being taken with reference to the plane of y,z, we shall obtain

$$Mx_{\prime}=\int \pi y^2 x dx \ldots\ldots (100);$$

and by dividing this equation by the preceding, we find

$$x_{\prime}=\frac{\int y^2 x dx}{\int y^2 dx} \ldots\ldots (101).$$

We must eliminate one of the variables in this formula, by means of the equation of the curve, and then integrate between the limits $x=\text{AP}$ and $x=\text{AQ}$.

189. This formula being applied to the determination of the centre of gravity of a cone, it will be necessary to obtain the two integrals

$$\int y^2 dx \text{ and } \int xy^2 dx.$$

Eliminating y^2 by the equation of the generatrix $y=ax$, we obtain, after integration,

$$\int y^2 dx=\int a^2x^2 dx=\frac{a^2x^3}{3},$$

$$\int y^2 x dx=\int a^2x^3 dx=\frac{a^2x^4}{4}.$$

There are no constants introduced by integration, since the volume is equal to zero at the origin A (*Fig.* 88). These values, being substituted in the formula (101), give

$$x_{,}=\frac{\frac{a^2x^4}{4}}{\frac{a^2x^3}{3}}=\tfrac{3}{4}x:$$

from which we conclude that *the centre of gravity of a cone is at a distance from the vertex equal to three-fourths of the altitude* Ax.

190. As a second example, let the required centre of gravity be that of the volume of a *paraboloid* generated by the revolution of the parabolic arc AM (*Fig.* 85) about the axis Ax. The equation of the curve being $y^2=px$, we have

$$\textstyle\int y^2dx=\int pxdx=\frac{1}{2}px^2,$$

$$\textstyle\int y^2xdx=\int px^2dx=\frac{1}{3}px^3:$$

these values substituted in formula (101), give

$$x_{,}=\frac{\frac{1}{3}px^3}{\frac{1}{2}px^2}=\tfrac{2}{3}x.$$

The constants introduced by integration are equal to zero in the present instance, for the reasons assigned in the preceding paragraph.

191. Let the solid of revolution be an *ellipsoid*, the equation of whose generatrix is

$$y^2=\frac{b^2}{a^2}(a^2-x^2):$$

this value of y^2 being substituted in the integrals of equation (101), we obtain, since the constants are equal to zero,

$$\int y^2dx=\frac{b^2}{a^2}\int(a^2dx-x^2dx)=\frac{b^2}{a^2}\left(a^2x-\frac{x^3}{3}\right),$$

$$\int y^2xdx=\frac{b^2}{a^2}\int(a^2xdx-x^3dx)=\frac{b^2}{a^2}\left(\frac{a^2x^2}{2}-\frac{x^4}{4}\right).$$

These values reduce equation (101) to

$$x_{,}=\frac{\frac{1}{2}a^2x-\frac{1}{4}x^3}{a^2\ -\frac{1}{3}x^2}=\frac{6a^2x-3x^3}{12a^2-4x^2};$$

and by taking the integral between $x=0$ and $x=a$, we find, for the absciss of the centre of gravity of the semi-ellipsoid,

$$x_{,}=\tfrac{3}{8}a.$$

192. *To find the centre of gravity of a volume generated*

by the revolution of an area embraced by a curve BMCM′ (*Fig.* 89) *about the axis of x, this axis being situated entirely without the curve.*

Represent by y and y' the ordinates MP and M′P: the volume generated by the revolution of the element Mm′, will be equal to the difference of the volumes generated by the elementary rectangles Mp and M′p; the expressions for these volumes being $\pi y^2 dx$ and $\pi y'^2 dx$, that of the element of the solid will be $\pi(y^2-y'^2)dx$; hence, if we denote by M the entire volume of the solid generated, we shall have

$$M=\pi\int(y^2-y'^2)dx.$$

By taking the moments with reference to the plane of y, z, we obtain

$$Mx_{,}=\pi\int(y^2-y'^2)xdx.$$

The value of $x_{,}$ will be alone necessary, since the centre of gravity must be situated on the axis of abscisses.

Of the Centrobaryc Method.

193. Let $x_{,}$ and $y_{,}$ represent the co-ordinates of the centre of gravity of a plane surface MPP′M′ (*Fig.* 90), the area of which is represented by λ. The moment of the element of this surface, taken with reference to the axis of x, is, by Art. 182, $\frac{1}{2}y\times ydx$; and by making the sum of the moments of all the elements equal to the moment of the whole body supposed concentrated at its centre of gravity, we have

$$\int\tfrac{1}{2}y^2dx=y_{,}\lambda.$$

The two members of this equation being multiplied by the quantity 2π, it becomes

$$\int\pi y^2dx=2\pi y_{,}\lambda:$$

The expression $\int\pi y^2dx$ represents the volume generated by the revolution of the given surface about the axis of x, and the second member $2\pi y_{,}\lambda$ is the product of the generating surface by the circumference described by the centre of gravity; hence, we deduce this general theorem: *The volume of every solid of revolution is equal to the product of the generating area by the circumference described by its centre of gravity.*

194. Let it be required, for example, to determine the

volume of the solid generated by the revolution of an isosceles triangle ABC (*Fig.* 91) about the axis of x. Denote CD by h, and AB by a; the generating area will then be expressed by $\frac{1}{2}ah$. But the centre of gravity of the generating triangle being at a distance from C equal to $\frac{2}{3}$CD, the circumference described by this point will be $\frac{1}{3}h\times 2\pi$. Hence, the volume will be expressed by the product $\frac{1}{3}h\times 2\pi\times\frac{1}{2}ah=\frac{1}{3}\pi ah^2$.

As a second example, let us determine the volume of a right cone generated by the revolution of the right-angled triangle ABC (*Fig.* 92) about the line AB. The area of the generatrix will be $\frac{1}{2}\text{AB}\times\text{BC}$. The line CE being drawn to the middle of the side AB, the centre of gravity G of the generating area will be situated upon this line at a distance from the point E equal to $\frac{1}{3}$EC (Art. 172); its ordinate GD will therefore be determined by the proportion

$$3 : 1 :: \text{EC} : \text{EG} :: \text{CB} : \text{GD};$$

whence,

$$\text{GD}=\tfrac{1}{3}\text{CB}.$$

The path described by the centre of gravity will therefore be expressed by $\frac{2}{3}\pi\times\text{CB}$; which, multiplied by the area of the generating triangle gives the volume of the cone equal to $\frac{2}{3}\pi\times\text{CB}^2\times\frac{1}{2}\text{AB}=\frac{1}{3}\text{AB}\times\pi\times\text{CB}^2$.

195. Again, let the volume be that of a right cylinder: the ordinate GE of the centre of gravity of the generating rectangle (*Fig.* 93) being equal to $\frac{1}{2}$AC, the path described by this point will be πAC. This expression being multiplied by the generating area which is equal to $\text{AB}\times\text{AC}$, we have $\pi\times\text{AC}^2\times\text{AB}$ for the volume of the cylinder.

196. The area of any surface of revolution may be found by a rule analogous to the preceding. For, if we consider the surface generated by the revolution of any curve MN (*Fig.* 94) about the axis of abscisses, and denote by y_1 the ordinate of its centre of gravity G, we shall have, by Art. 178,

$$\int y\sqrt{(dx^2+dy^2)}=y_1\times\text{arc MN} \;.\;.\;.\;.\;.\; (102);$$

and by multiplying each member by 2π, this equation becomes

$$\int 2\pi y\sqrt{(dx^2+dy^2)}=2\pi y_{,}\times \text{arc MN}.$$

The expression $\int 2\pi y\sqrt{(dx^2+dy^2)}$ representing the area of the surface generated, we conclude, that *the area of a surface of revolution is equal to the product of the generating arc by the circumference described by its centre of gravity.*

197. Thus, to determine the surface of a conic frustrum generated by the revolution of the right line CD (*Fig.* 95) about the axis of x; we have the ordinate EG of the centre of gravity equal to $\frac{AC+DB}{2}$; and $2\pi\times\frac{AC+DB}{2}$ equal to the circumference described by this point: hence, the product of this expression by the length of the generatrix CD gives $2\pi\times\frac{AC+BD}{2}\times CD=2\pi\,.\,GE\,.\,CD$ for the convex surface of the conic frustrum.

198. The two preceding theorems may be included in a single enunciation, viz.: *Every solid or surface of revolution is equal to the product of its generatrix by the circumference described by the centre of gravity of the generatrix.*

Machines.

199. Machines serve to transmit the action of forces in directions different from those in which the forces are applied, and to modify the effects of those forces.

The force applied to a machine is called the power, and that which tends to oppose the effect of the power is called the resistance.

The most simple machines are the cord, the lever, and the inclined plane. To these are sometimes added the pulley, the wheel and axle, the screw, and the wedge, which may be formed by very simple combinations of the first three. These machines are usually called the *Mechanical Powers.*

Cords.

200. We shall adopt the hypothesis that cords are perfectly flexible, that they are inextensible, without weight, and reduced to their axes. If the extremities of a cord be solicited

by two equal forces P and Q (*Fig.* 96), which tend to stretch it, the tension of the cord will be measured by one of these forces; for, since the equilibrium subsists, we may regard A, the middle of the line PQ, as a fixed point, and drop the consideration of that portion of the cord included between A and Q; thus, the force P, acting alone against the fixed point A, will measure the tension of the cord PQ.

201. When the force Q exceeds P, a portion of Q equal to P is employed to stretch the cord, while the remaining part of the force tends only to move the cord in the direction from P towards Q: thus the tension will be measured by the least of these forces.

202. If three cords be united by a knot, the conditions of equilibrium are similar to those which obtain when any three forces act on a point. The force acting in the direction of each cord must be equal and directly opposed to the resultant of the other two; hence, the conditions of equilibrium require that the three forces be situated in the same plane, and bear to each other the following relations (*Fig.* 97),

$$P : Q : R :: \sin p : \sin q : \sin r.$$

203. This proportion will be insufficient to establish the equilibrium, if the cords are united by a sliding knot. For, by regarding P and R as fixed points (*Fig.* 98), to which the cord PCR is attached, if the force Q be supposed to act upon this cord by means of a ring or sliding knot, the point C will describe an ellipse, the plane of which will pass through the points P and R. But the revolution of this ellipse around the axis PR will generate an ellipsoid, having its transverse axis equal to PC+CR, and the point C will necessarily be found upon the surface of the ellipsoid, or, in other words, at some point of the moveable ellipse; but the point C being only subject to motion when the force Q has a component in the direction of the elliptical arc, the equilibrium will be maintained when the direction of the force Q is normal to the ellipse. If the line T*t* be drawn tangent to the curve, we shall have, from the well known property of the ellipse,

$$\angle TCP = \angle RCt;$$

and by subtracting these angles from the right angles TCN, *t*CN, there will remain

$$\angle PCN = \angle NCR;$$

thus the angle PCR must be bisected by the direction of the force Q, and the proportion

$$P : R :: \sin NCR : \sin PCN$$

becomes, in the present case,

$$P : R :: \sin NCR : \sin NCR;$$

whence, P and Q are equal to each other.

204. The funicular machine consists of a number of cords united to each other at several knots, and maintaining an equilibrium between the forces applied to these cords.

205. When several forces P, R, S, T, &c. (*Fig.* 99), act conjointly at a single knot, their number will be reduced by unity, if we substitute for any two forces P and R their resultant R′; and by a repetition of the same process the entire system may always be reduced to three forces united at a single knot.

206. Let there be several forces P, P', P'', P''', P^{iv}, &c. (*Fig.* 100), acting at the knots A, B, C, &c. of the cord ABC. The conditions of equilibrium of these forces may be reduced to those of a system acting on a single point: for, let R represent the resultant of the forces P and P′; since its effect must be destroyed by the third force acting in the line AB, the direction of this resultant must coincide with the prolongation of AB: but the point of application of a force may be assumed any where on its line of direction, and hence we may transfer the force R to the point B. If it be there decomposed into two components parallel and equal to P and P′, the effect will be the same as if the two forces P and P′ had been transported parallel to their original directions, and applied at the point B. In like manner, by transporting the forces P, P', P'', &c., which are supposed to be applied at B, to the point C, the entire system may be considered as acting on this point. Thus the conditions of equilibrium are, (Art. 54),

$$\Sigma(P \cos\alpha)=0, \quad \Sigma(P \cos\beta)=0, \quad \Sigma(P \cos\gamma)=0.$$

To determine the ratio of the extreme tensions P and P_n, we will denote by t and t' the tensions of the portions AB and BC, and by

a the angle PAP′, a' the angle ABP″, a'' the angle BCP‴,
b the angle P′AB, b' the angle P″BC, b'' the angle P‴CP$^{\text{IV}}$;
we shall then obtain, Art. 202,

$$P : t :: \sin b : \sin a,$$
$$t : t' :: \sin b' : \sin a',$$
$$t' : P^{\text{IV}} :: \sin b'' : \sin a'';$$

whence, by multiplication, suppressing the factors which are common to the two first terms, we have

$$P : P^{\text{IV}} :: \sin b \times \sin b' \times \sin b'' : \sin a \times \sin a' \times \sin a''.$$

We may, in like manner, determine the relations between any other two forces.

207. If the forces P′, P″, P‴, &c. be supposed parallel, we shall have

$$b + a' = 180°, \quad b' + a'' = 180°;$$

and since the sine of an angle is equal to the sine of its supplement, we must have

$$\sin b = \sin a', \quad \sin b' = \sin a'';$$

and the preceding proportion will then reduce to

$$P : P^{\text{IV}} :: \sin b'' : \sin a.$$

If the forces P′, P″, and P‴ represent weights (*Fig.* 101), the entire system will be situated in the same vertical plane; for, the right line AP′ being vertical, the plane of the forces P, P′, and t will be vertical. For a similar reason, the plane of the forces t, P″, and t', will be vertical; but the line AB not being vertical, it is impossible to pass more than one vertical plane through it; hence, the forces P, P′, t, P″, and t' will be situated in the same vertical plane. The same reasoning may be extended to a greater number of forces.

208. The extreme forces P and P$^{\text{IV}}$ being required to sustain the resultant of all the others, this resultant must be directly opposed to that of the forces P and P$^{\text{IV}}$, and must consequently pass through the point G, at which the directions of those forces intersect. Moreover, its direction must be vertical, being parallel to the components P′, P″, and P‴, and it will therefore be represented by the vertical line GH drawn through the point G.

209. If we regard a heavy cord as a funicular polygon,

loaded with an infinite number of small weights, it results from what precedes that the effect produced on the fixed points by the weight of the cord may be estimated by drawing the tangents PG and QG (*Fig.* 102), and applying at G a weight equal to that of the cord; since if we denote this weight by G, we shall then have

$$P : Q : G :: \sin LGQ : \sin LGP : \sin PGQ.$$

Of the Catenary.

210. The catenary is the curve which a perfectly flexible cord assumes when it is suspended from two fixed points A and B (*Fig.* 103), and subjected to the action of the force of gravity. We will suppose that the cord is uniformly heavy, and that the force of gravity is exerted on every particle: it will readily appear, as in Art. 207, that the curve will be situated in a vertical plane. Let the origin of co-ordinates be assumed at A, the horizontal line AC being the axis of abscisses; the co-ordinates of a point M will then be $AP=x$, and $PM=y$. Through the point M, and through the origin A, let tangents AH and MH be respectively drawn, intersecting at the point H, and through this point draw the vertical line HL. If we consider the portion of the cord MA, we shall have, by Art. 209,

$$\textit{tension at A} : \textit{weight of the portion } AM :: \sin LHM : \sin AHM \ldots\ldots (103).$$

Let s denote the length of the arc AM; A the tension of the cord at the point A, which is exerted in the direction of the tangent AH; and α the angle included between this tangent and the horizontal line AC. The quantities A and α will remain constant.

The tension at A, being a quantity of the same kind as that contained in the second term of the preceding proportion, will necessarily be expressed by a weight; and if we represent by p the weight of a portion of the cord whose length is equal to unity, sp will express the weight of the part AM, and the tension at A will be of the form ap. Thus the two first terms in the above proportion will be replaced by the ratio $ap : sp$, or by its equal $a : s$; hence,

$$a : s :: \sin LHM : \sin AHM \ldots\ldots (104).$$

211. To determine the analytical expressions for the sines which enter into this proportion, we remark, that in the elementary triangle mMn, we have

$$Mm \times \sin mMn = mn, \quad Mm \times \cos mMn = Mn;$$

or,

$$\sin mMn = \frac{mn}{Mm}, \quad \cos mMn = \frac{Mn}{Mm};$$

and replacing these elementary lines by their analytical values, these equations become

$$\sin mMn = \frac{dx}{ds}, \quad \cos mMn = \frac{dy}{ds} \; \ldots\ldots \; (105).$$

But the angle mMn included between the vertical and the arc of the curve, is equal to the angle LHK formed by the vertical with the tangent at M; hence,

$$\sin LHK = \frac{dx}{ds}, \quad \cos LHK = \frac{dy}{ds} \; \ldots\ldots \; (106).$$

The first of these equations may be reduced to

$$\sin LHM = \frac{dx}{ds} \; \ldots\ldots \; (107);$$

for the angles LHK and LHM being supplements of each other, we have

$$\sin LHK = \sin LHM.$$

Again, the angles AHK and AHM being supplements of each other, we obtain

$$\sin AHM = \sin AHK = \sin (LHK - LHA);$$

and from the well known trigonometrical formula for the sine of the difference of two angles, we have

$$\sin AHM = \sin LHK \cos LHA - \sin LHA \cos LHK;$$

eliminating sin LHK and cos LHK by means of the equations (106), we find

$$\sin AHM = \frac{dx}{ds} \cos LHA - \frac{dy}{ds} \sin LHA \; \ldots\ldots \; (108).$$

The triangle LAH being right-angled at L, the angles LHA and HAL are complements of each other, and the latter having been denoted by α, we obtain

$$\cos LHA = \sin \alpha, \quad \sin LHA = \cos \alpha.$$

212. These values substituted in equation (108) give

$$\sin \mathrm{AHM} = \frac{dx}{ds}\sin\alpha - \frac{dy}{ds}\cos\alpha \;.\;.\;.\;.\;.\; (109);$$

and the equations (107) and (109) convert the proportion (104) into

$$a : s :: \frac{dx}{ds} : \frac{dx}{ds}\sin\alpha - \frac{dy}{ds}\cos\alpha.$$

From this proportion we deduce the equation

$$s = a\sin\alpha - a\frac{dy}{dx}\cos\alpha \;.\;.\;.\;.\;.\; (110).$$

This equation contains three variables, one of which may be eliminated by means of the relation

$$ds = \sqrt{(dx^2 + dy^2)}.$$

For, by differentiating equation (110), regarding dx as constant, we find

$$ds = -a\cos\alpha\frac{d^2y}{dx};$$

and by equating these values of ds, and dividing each member of the equation by dx, we obtain

$$\sqrt{\left(1+\frac{dy^2}{dx^2}\right)} = -a\cos\alpha\frac{d^2y}{dx^2},$$

or, by division,

$$1 = \frac{-a\cos\alpha\dfrac{d^2y}{dx^2}}{\sqrt{\left(1+\dfrac{dy^2}{dx^2}\right)}},$$

This equation will become integrable, if we multiply its two members by $2dy$; we shall thus obtain

$$2dy = -a\cos\alpha\frac{2dy\dfrac{d^2y}{dx^2}}{\sqrt{\left(1+\dfrac{dy^2}{dx^2}\right)}};$$

whence, by integration,

$$y = -a\cos\alpha\sqrt{\left(1+\frac{dy^2}{dx^2}\right)} + c.$$

This equation being multiplied by dx gives

$$(c-y)dx = a\cos\alpha\sqrt{(dx^2+dy^2)};$$

and by reduction

$$\frac{dy}{dx}=\frac{\sqrt{[(c-y)^2-a^2\cos^2\alpha]}}{a\cos\alpha}\ \ldots\ldots\ (111).$$

213. The constant c may be determined by the consideration that at the point A,

$$s=0,\quad y=0,\quad \text{and } \frac{dy}{dx}=\text{tang}\,\alpha.$$

These values reduce the equation (111) to

$$\text{tang}\,\alpha=\frac{\sqrt{(c^2-a^2\cos^2\alpha)}}{a\cos\alpha};$$

from which we deduce

$$a\,\text{tang}\,\alpha\cos\alpha=\sqrt{(c^2-a^2\cos^2\alpha)}:$$

but

$$\text{tang}\,\alpha\cos\alpha=\sin\alpha;$$

whence,

$$a^2\sin^2\alpha=c^2-a^2\cos^2\alpha,$$

and consequently

$$c^2=a^2\,(\sin^2\alpha+\cos^2\alpha)=a^2.$$

Thus the constant c is equal to a, and by substituting its value in equation (111), we find for the differential equation of the catenary,

$$\frac{dy}{dx}=\frac{\sqrt{[(a-y)^2-a^2\cos^2\alpha]}}{a\cos\alpha}\ \ldots\ldots\ (112).$$

214. It appears from a comparison of this equation with (110), that the catenary curve is rectifiable; for, if the preceding value of $\frac{dy}{dx}$ be substituted in equation (110), we shall obtain

$$s=a\sin\alpha-\sqrt{[(a-y)^2-a^2\cos^2\alpha]}\ \ldots\ldots\ (113):$$

from this expression the value of s may be readily found in terms of y, when the constants a and α have been determined.

215. To integrate the differential equation of the catenary, we make

$$a-y=z,\quad a\cos\alpha=b\ \ldots\ldots\ (114);$$

and we then obtain

$$dy=-dz;$$

these values substituted in equation (112) give

$$dx=-\frac{bdz}{\sqrt{(z^2-b^2)}}\ldots\ldots(115);$$

this expression becomes integrable by making

$$\sqrt{(z^2-b^2)}=z-t\ldots\ldots(116):$$

which by squaring and reducing, gives

$$2zt=b^2+t^2.$$

By the differentiation of this equation, we obtain

$$zdt+tdz=tdt,$$

or,

$$\frac{dz}{z-t}=-\frac{dt}{t}.$$

This relation, in connexion with that assumed above (116), converts the equation (115) into

$$dx=\frac{bdt}{t},$$

which gives, by integration,

$$x=b\log t+e;$$

and by substituting for t its value expressed in terms of z, we obtain

$$x=b\log[z-\sqrt{(z^2-b^2)}]+e;$$

or finally, by replacing the quantities b and z, by their values given in equations (114), we find

$$x=a\cos\alpha\log\{a-y-\sqrt{[(a-y)^2-a^2\cos^2\alpha]}\}+e\ldots.(117).$$

216. To determine the value of the constant e, we observe that at the point A, $x=0$, and $y=0$; which conditions reduce the equation (117) to

$$e=-a\cos\alpha\log\{a[1-\sqrt{(1-\cos^2\alpha)}]\}.$$

This value substituted in equation (117) gives

$$x=a\cos\alpha\log[a-y-\sqrt{(a-y)^2-a^2\cos^2\alpha}]$$
$$-a\cos\alpha\log[a(1-\sqrt{1-\cos^2\alpha})];$$

or by reduction,

$$x=a\cos\alpha\log\left(\frac{a-y-\sqrt{[(a-y)^2-a^2\cos^2\alpha]}}{a[1-\sqrt{(1-\cos^2\alpha)}]}\right)\ldots\ldots(118).$$

Such is the equation of the catenary.

217. The values of the constants c and e have been determined in functions of a and α; but these two quantities are

still unknown. To determine their values, we will suppose that x' and y' represent the known co-ordinates of the second point of suspension B, and l the length of the curve AMB; these values being substituted in the equations (113) and (118), we obtain

$$l=a\sin\alpha-\sqrt{[(a-y')^2-a^2\cos^2\alpha]},$$

$$x'=a\cos\alpha\log\left(\frac{a-y'-\sqrt{[(a-y')^2-a^2\cos^2\alpha]}}{a[1-\sqrt{(1-\cos^2\alpha)}]}\right).$$

218. These equations, in connexion with the relation

$$\cos^2\alpha+\sin^2\alpha=1,$$

determine the values of a, $\cos\alpha$, and $\sin\alpha$, in functions of x', y', and l. But another difficulty still presents itself; this consists in the proper choice of the signs with which to affect $\cos\alpha$, and the radicals which in the preceding expressions have not received the double sign. To resolve this difficulty, we will determine the co-ordinates of that point to which the maximum ordinate appertains. The characteristic property of this point is that $\frac{dy}{dx}=0$, which reduces equation (112) to

$$\frac{\sqrt{[(a-y)^2-a^2\cos^2\alpha]}}{a\cos\alpha}=0;$$

and consequently,

$$a-y=a\cos\alpha\;.\;.\;.\;.\;.\;(119).$$

To establish the condition that this equation belongs to a maximum, rather than to a minimum value, we attribute the proper sign to the second differential co-efficient $\frac{d^2y}{dx^2}$.

But by squaring the equation (112), we obtain

$$\frac{dy^2}{dx^2}=\frac{(a-y)^2-a^2\cos^2\alpha}{a^2\cos^2\alpha};$$

and by differentiating, and dividing each member by $2dy$, we find

$$\frac{d^2y}{dx^2}=-\frac{a-y}{a^2\cos^2\alpha};$$

substituting in this equation the value of $a-y$ determined in equation (119), we obtain

$$\frac{d^2y}{dx^2}=-\frac{1}{a\cos\alpha},$$

219. This equation indicates that the condition of a maximum will be fulfilled by attributing the same sign to a and $\cos \alpha$; but these signs must be positive; for, if they were negative, the value of y determined by the equation (119) would be also negative, which is evidently inadmissible in the hypothesis adopted, that the positive ordinates are reckoned from the line AC downwards. From the equation (119) we likewise infer, that the quantity a exceeds the maximum value of y, and therefore that it exceeds all other values. Let EF represent the maximum ordinate (*Fig.* 103); it is evident that between the limits $x=0$ and $x=\mathrm{AE}$, as y increases, the arc of the catenary will likewise increase. But it appears from equation (113) that the increase of y will not necessarily involve that of the arc s, unless the radical in that formula be affected with the negative sign. For, as y increases, the quantity $a-y$ will decrease, and the value of the radical will therefore decrease; but the smaller the value of this radical, the less it will diminish the positive part of the expression $a \sin \alpha$, and the greater will be the value of the arc. The equation (113) is therefore in perfect accordance with the hypothesis that the co-ordinate has not attained its maximum value. But from $x=\mathrm{AE}$ to $x=\mathrm{AD}$, the arc s should increase while y diminishes, and since this decrease in the value of y augments the value of the radical expression, the required condition can only be fulfilled by affecting the radical with the positive sign: thus, between the limits $x=\mathrm{AE}$ and $x=\mathrm{AD}$, the sign of the radical must be changed in the formula (113).

Of the Lever.

220. The lever is a bar of wood or metal moveable around a fixed point, which is called the fulcrum. To simplify the considerations which relate to this machine, we shall regard the lever as destitute of thickness, and will therefore represent it by a simple line, either straight or curved. Let a lever AB (*Fig.* 104) be solicited by the two forces P and P′; the effect of these forces cannot be destroyed by the resistance of a fixed

point C, unless they are situated in a plane passing through this point. If this condition be fulfilled, the equilibrium will be maintained, when the sum of the moments taken with reference to the point C is equal to zero.

221. If the lever is capable of sliding along its point of support, it will also be necessary that the resultant of the forces acting on the lever should be perpendicular to the lever at the point of support.

222. When the lever is straight and the two forces parallel to each other, if p and p' represent the lengths of the portions AC and BC (*Fig.* 106), we shall have from the theory of parallel forces (Art. 73),

$$P : P' :: p' : p;$$

from which we infer, that *when the forces are in equilibrio, their intensities will be inversely proportional to the arms of the lever.*

223. If the lever be curved, and a right line ED (*Fig.* 105) be drawn through the fulcrum C, the forces may be conceived to be applied at the points E and D taken on their respective directions; we shall thus obtain

$$P : P' :: CD : CE.$$

224. Levers are divided into three kinds. In the first kind, the fulcrum C (*Fig.* 106) is situated between the power and the resistance: in the second kind, the resistance R (*Fig.* 107) is situated between the power and the fulcrum; and in the third kind (*Fig.* 108), the power is between the fulcrum and the resistance.

The balance and steelyard are examples of the first kind of lever; a bar of iron used in raising weights and having its fulcrum at one extremity, forms a lever of the second kind; the treddle of a turning lathe is a lever of the third kind.

225. The effect produced by the weight of a lever may be readily estimated by regarding it as a force S applied at the centre of gravity of the lever. For example, let P and P′ (*Fig.* 109) be two weights suspended from the extremities of the lever AB, whose centre of gravity is situated at G; we

shall have, by virtue of the principle of the moments,

$$P' \times CB + S \times CG = P \times AC.$$

This equation will determine either P or P'; and the weight sustained by the fixed point will be

$$P + P' + S.$$

If the power and resistance act in opposite directions, regard must be had to the directions in which they tend to turn the lever; thus, in *Fig.* 110, the equation of the moments becomes

$$P \times CA + S \times CG = P' \times CB \,.\,.\,.\,.\,.\, (120);$$

and the weight sustained by the fulcrum is

$$P + S - P'.$$

226. Let the lever CB (*Fig.* 110) be supposed homogeneous, and of uniform weight throughout its length: represent by m the weight of a portion of the lever whose length is one foot. If x represent the length of the lever expressed in feet, its weight S will be expressed by mx, and should be regarded as a force acting at its centre of gravity, which corresponds to the middle point G: thus, if we make $CA = a$, the equation (120) will then become

$$Pa + \tfrac{1}{2}x \times mx = P' \times x;$$

from which we deduce

$$P' = \frac{Pa}{x} + \tfrac{1}{2}mx \,.\,.\,.\,.\,.\, (121).$$

If, therefore, x be assumed arbitrarily, this formula will make known the value of P'; but it may be required to assign the value of x which shall render P' the least possible; we must then regard P' as a function of x, and make the differential co-efficient $\frac{dP'}{dx}$ equal to zero; we shall thus obtain

$$\frac{dP'}{dx} = -\frac{Pa}{x^2} + \tfrac{1}{2}m = 0;$$

whence,

$$x^2 = \frac{2Pa}{m}, \text{ and } x = \sqrt{\left(\frac{2Pa}{m}\right)}.$$

By substituting this value in equation (121), we obtain

$$P'=\frac{Pa}{\sqrt{\left(\frac{2Pa}{m}\right)}}+\tfrac{1}{2}m\sqrt{\left(\frac{2Pa}{m}\right)};$$

or, by reduction,

$$P'=\frac{2Pa}{\sqrt{\left(\frac{2Pa}{m}\right)}}=\sqrt{(2Pam)}.$$

227. *The common balance* is an important application of the lever. It consists essentially of a lever having equal arms, from the extremities of which are suspended scales of equal weight. The lever of the balance, which is called the *beam*, is sustained by a horizontal axis perpendicular to its length, which rests upon a firm support, and the substance to be weighed, being introduced into one of the scales, is counterpoised by the addition of known weights in the opposite scale. The figure of the beam is so chosen that its centre of gravity will be found immediately beneath the axis, or centre of motion, when the beam has assumed a horizontal position; and the weights suspended from its two extremities are known to be equal when they will retain the beam in this situation. If the centre of gravity were found upon the axis, the beam would obviously rest in any position, and there would be nothing to indicate the equality of the weights in the two scales; and if this centre were situated above the axis, the beam would have a tendency to overturn if deranged in the slightest degree from the horizontal position.

228. When the balance has been constructed with such accuracy that the lengths of the arms are exactly equal, the beam will assume the horizontal position if equal weights be introduced into the two scales; but in the false balance, where the lengths of the arms are unequal, the weights necessary to maintain the beam in this position are likewise unequal. In this case, the weight of the body may be obtained by counterpoising it successively in the two scales: *the true weight will be a geometrical mean between the two apparent weights.* For let p and p' represent the lengths of the two arms, and W the true weight of the body. Then, if a weight P, suspended from the extremity of the arm p, be supposed to sustain the weight W when suspended from the extremity

of the arm p', the conditions of equilibrium in the lever (Art. 220) will give

$$Pp=Wp'.$$

But if the weight W be transferred to the extremity of the arm p, it will be necessary to apply a different weight P′ to the extremity of the arm p', in order that the equilibrium may be preserved. Thus we shall have

$$P'p'=Wp;$$

and by multiplying the corresponding members of these two equations, we obtain

$$PP'pp'=W^2pp';$$

or, by reduction,

$$W=\sqrt{(PP')};$$

hence, the truth of the proposition enunciated becomes apparent.

229. It is frequently necessary that the balance employed should possess great *sensibility*, or should be capable of indicating very minute differences in the weights of the substances placed in the two scales. The sensibility of the balance is measured by the smallness of the weight necessary to produce a given inclination of the beam, when the scales are charged with a given load.

The sensibility depends upon the following particulars.

1°. The beam should be as light as is consistent with a proper degree of strength, in order that the friction at the axis, which is proportional to the pressure, may oppose the least possible resistance to the motion of the beam.

For the same reason the axis is constructed of hardened steel, and has the form of a knife-edge, or triangular prism, the lower edge of which rests upon polished steel or agate planes.

2°. The lengths of the arms should be as great as possible, other things remaining the same, since the moments of the weights introduced into the scales, taken with reference to the centre of motion, will be directly proportional to these lengths. Thus, the same weight, placed at twice the distance from the centre of motion, will exert a double effort to turn the beam.

3°. The sensibility will be increased by diminishing the distance between the centre of gravity of the beam and the centre of motion. For, when the beam has been deranged from the horizontal position through a given angle (*Fig.* 111), the weight of the beam W, which acts at its centre of gravity G, will exert an effort to restore it to its former position, which effort will be directly proportional to the moment of the weight W, taken with reference to the centre of motion D; this moment will be expressed by $W \times dg$. But the derangement of the beam having been made through a given angle, the distance dg will evidently be proportional to DG, the distance between the centre of gravity of the beam and the centre of motion. Thus, in proportion as the distance DG is diminished, the tendency of the weight of the beam to counteract the derangement which would be produced by an inequality of the weights in the two scales will likewise be diminished, or the sensibility will be increased.

4°. The line joining the points of suspension of the two scales should pass through the centre of motion. For, if the centre of motion be found at C above the line AB, and the beam be supposed to have assumed the inclined position represented in *Fig.* 111, the effective arm of lever CE′ of the scale P′ will evidently be greater than the arm CE of the scale P. Thus the beam may have a tendency to return to the horizontal position, although the weight P′ be less than P. And if, on the contrary, the centre of motion be placed at a point C′ below the line AB, the lever-arm C′F of the scale P will exceed that of the scale P′, and the beam would therefore have a tendency to overturn, although the weights in the scales were equal to each other. When the centre of motion is situated at the point D, the equality of the two arms will be preserved, whether the beam be in a horizontal or inclined position.

5°. The sensibility of the balance will be increased by diminishing the load with which the scales are charged, since the friction at the axis will be diminished in the same proportion.

230. A very accurate balance will be sensibly affected by the addition of $\frac{1}{100000}$ part of the load with which the scales are charged.

231. The *steelyard*, represented in *Fig.* 112, is a balance having unequal arms, and is so constructed that a moveable weight P, applied successively at different points of the longer arm, shall sustain in equilibrio different weights suspended from the extremity of the shorter arm. The longer arm GB is so graduated as to indicate the weight which will be supported by the moveable weight P, when placed at each of these divisions.

232. To discover the law according to which this arm should be graduated, we will denote by

W, W', W'', &c., the weights suspended successively from the extremity of the shorter arm,

p, p', p'', &c., the corresponding distances at which the weight P must be placed to maintain the equilibrium,

r, the length of the shorter arm,

w, the weight of the beam,

r', the distance of its centre of gravity from the fulcrum.

Then, if the centre of gravity of the beam be supposed to lie on the side of the longer arm, as usually happens, the conditions of equilibrium will give

$$\begin{aligned} Wr &= wr' + Pp, \\ W'r &= wr' + Pp', \\ W''r &= wr' + Pp'', \\ &\text{\&c.} \quad \text{\&c.} \quad \text{\&c.}; \end{aligned}$$

and by subtracting each of these equations from that which follows, we obtain

$$\begin{aligned} (W'-W)r &= (p'-p)P, \\ (W''-W')r &= (p''-p')P, \\ (W'''-W'')r &= (p'''-p'')P. \end{aligned}$$

If the weights W, W', W'', &c. be supposed to increase in arithmetical progression, we shall have

$$W'-W=W''-W'=W'''-W''=\text{\&c.};$$

and therefore

$$p'-p=p''-p'=p'''-p''=\text{\&c.};$$

thus the distances p, p', p'', &c. will likewise increase in arithmetical progression.

If, for example, the moveable weight P when placed at a point F should be found to support a weight of 10 pounds, and if when placed at the point E, the weight supported should be found equal to 20 pounds, we might divide the distance EF into ten equal parts, and the points of division will correspond to the weights 11 pounds, 12 pounds, 13 pounds, &c.

The zero of the scale will evidently be found at that point from which the weight P is suspended when it merely serves to counterpoise the weight of the lever. The steelyard is frequently constructed in such a manner that the two arms of the lever counterpoise each other: the zero of the scale will then coincide with the fulcrum.

Of the Pulley.

233. The pulley is a wheel having a groove cut in its circumference for the purpose of receiving a cord which partially envelopes it: when a motion is imparted to this cord it is immediately communicated to the pulley, causing it to turn about an axis which passes through its centre, and is usually supported by a curved piece of iron terminating in a hook (*Fig.* 113).

Pulleys are distinguished into two kinds, the fixed and the moveable. In the fixed pulley, the hook is attached to an immoveable point, as in (*Fig.* 113); and in the moveable pulley the resistance R (*Fig.* 114) is applied to the hook.

234. The conditions of equilibrium in the fixed pulley require the equality of the power P, and the resistance Q (*Fig.* 113); for, if the intensities of these forces were unequal, the greater of the two would prevail.

This property may also be demonstrated in the following manner: we prolong the directions of the two forces which act tangentially, until they intersect at the point E; their resultant will pass through this point; and since the effect of this resultant is destroyed by the resistance of the axis of the pulley at O, the resultant must likewise pass through this point. But the triangles EP′O, EQ′O being identical, the angle P′EQ′ is bisected by the direction of the resultant; whence it follows that the force P is equal in intensity to the force Q.

235. Let there be now taken the equal parts E*g* and E*h*,

and construct the parallelogram Egfh; the forces P and Q being represented by the lines Eg and Eh, their resultant R will be represented by Ef: we shall thus have the proportion

$$P : Q : R :: Eg : Eh : Ef;$$

and from the similarity of the triangles Egf and P′OQ′, whose sides are respectively perpendicular to each other, we obtain

$$P'O : OQ' : P'Q' :: Eg : Eh : Ef;$$

hence,

$$P : Q : R :: P'O : OQ' : P'Q':$$

from which we conclude, that *in the fixed pulley, each of the forces is to the resultant, or the pressure upon the point of support, as the radius of the pulley to the chord of the arc with which the rope is in contact.*

The equality of the forces P and Q having been demonstrated, it follows that the advantage of the fixed pulley consists only in changing the direction of the power.

236. Let the cord QABP (*Fig.* 114) be supposed to embrace the arc AB of a moveable pulley, one extremity of the cord being attached to the fixed point Q; and let a power P be applied to the other extremity, for the purpose of sustaining a resistance R. The reaction exerted by the fixed point Q will be similar in its effect to a force Q, and the conditions of equilibrium between P, Q, and R will be the same as in the case of the fixed pulley, except that the resistance which was then denoted by Q, will in the present case be represented by R. Thus, the relation between the power and resistance will be determined from the proportion.

P : R :: *radius* : *chord of the arc* AB.

As the intensity of the power may be less than that of the resistance, the moveable pulley may effect a gain of power.

When the cords are parallel, the preceding proportion becomes

P : R :: *radius* : *diameter* :: 1 : 2,

and the power is then equal to one-half the resistance.

If the chord of the arc be equal to the radius, the power and resistance will become equal; and when the radius exceeds the chord, the use of the moveable pulley will induce a loss of power.

237. By the combination of a number of moveable pulleys we may succeed in raising enormous weights by the application of a very small force: the pulleys may be arranged in the following manner:

The weight R (*Fig*. 115) is suspended from the hook of the moveable pulley ABD, around which a cord is passed having one of its extremities attached to the fixed point K, and the other to the hook of the pulley A′B′D′. This second pulley is in like manner supported by a cord, attached at one end to the point K′, and at the other to the hook of the pulley A″B″D″; and the same arrangement is continued to the last pulley, which is embraced by a cord connected at one end with a fixed point K″, the force P being applied to the other. If an equilibrium subsists throughout the system, the tensions of the cords AE, A′E′, &c. being denoted by T, T′, &c., we shall have, by supposing there are three pulleys,

$$R : T :: AB : AC,$$
$$T : T' :: A'B' : A'C',$$
$$T' : P :: A''B'' : A''C''.$$

These proportions being multiplied together give

$$R : P :: AB \times A'B' \times A''B'' : AC \times A'C' \times A''C'';$$

from which we conclude, that the power is to the resistance as the continued product of the radii of the pulleys is to the continued product of the chords of the arcs embraced by the ropes.

When the ropes are parallel these chords become diameters, and the proportion is reduced to

$$R : P :: 2^3 : 1;$$

and, in general, for a number of pulleys denoted by n,

$$R : P :: 2^n : 1.$$

238. This arrangement of pulleys is seldom adopted, on account of its requiring too great a space. For, if the ropes be parallel, as represented in *Fig*. 116, and the centre of the pulley BOC be raised through a height denoted by h, the line BC being brought into the position bc, each branch of the

cord D″CBX must be shortened by the quantity $Bb=Cc=h$, and the whole rope will therefore be shortened by the quantity $2h$: consequently, the pulley AE will rise through the distance $2h$; for a similar reason, the third pulley will rise through a distance $4h$, equal to twice that described by the second; the same may be said of any number of pulleys: and the power P applied to the extremity of the last rope must rise through twice the distance which the last pulley ascends. Thus with a number of pulleys represented by n, the power will rise through a distance expressed by $2^n h$, and we therefore lose in the space described, in the same proportion that we gain in power.

To estimate the pressures sustained by the fixed points D, D′, D″, &c., we will represent them by Q, Q′, Q″; then, calling S and X the tensions of the cords SA and XB, we shall have

$$P=Q, \quad S=Q', \quad X=Q'';$$

which values substituted in the proportions

$$P:S::1:2,$$
$$S:X::1:2,$$

give

$$Q'=2P, \quad Q''=4P.$$

239. The *muffle* is a combination of several pulleys, all of which are disposed in the same block, and have a common cord passing around their respective circumferences.

To determine the relation between the power and the resistance in the muffle, represented in *Fig.* 117, we remark that the several branches of the rope must be equally stretched, and that these tensions acting conjointly must produce an equilibrium with the resistance R, which may therefore be regarded as solicited by six equal and parallel forces. The force Q will be measured by the intensity of one of these equal forces, and will consequently be equal to one-sixth of the resistance. Or, in general, *the power will be to the resistance as unity to the number of cords which support the resistance.*

240. In the use of either system of pulleys, a certain force will be necessary to overcome the weights of the moveable pulleys. The value of this force may be readily estimated

by regarding the weight of each pulley as an additional force applied to its hook. Thus, in the system with separate ropes represented in *Fig*. 116, the weight of the pulley BOC may be considered as applied to the hook, and will be equally supported by the cords BX and CD″: and since the addition of every moveable pulley reduces the power one-half, it follows, that the power will support one-half the weight of the upper pulley, one-fourth of the weight of AE, and one-eighth of the weight of BC. In the muffle (*Fig*. 117), the weight of the moveable block being equally distributed among the cords, the power will sustain one-sixth of this weight.

Of the Wheel and Axle.

241. This machine is composed of a wheel firmly connected with a cylindrical axis. To the circumference of the wheel a cord is attached, by means of which we can impart to it a motion of rotation, the effect of which is immediately communicated to the cylinder; a second cord being wrapped around the cylinder in a contrary direction, communicates motion to the resistance which is to be overcome. The axis is supported at its extremities by two cylindrical pivots which are of less diameter than the cylinder itself, and permit it to turn freely about the points of support.

242. To investigate the relation between the power and resistance in this machine, let us suppose its axis AB (*Fig*. 118) to have a horizontal position, and let a horizontal plane be drawn through this axis, intersecting the direction of the power P at the point F. Represent the intensity of the force P by the portion FP of its line of direction, and decompose it into two forces $FL = P'$ acting in a horizontal direction, and $FK = P''$ acting in a vertical direction. The direction of the force P' being prolonged will intersect the fixed axis, and the effect of this force will be destroyed by the reaction of the axis.

If motion be communicated by the force P, the point of application F of the vertical component P'' will descend, and the resistance R will ascend, while the point M, the intersection of the line HF with the axis of the cylinder, will remain immoveable. The point M may therefore be regarded as the fulcrum of a lever HF, to the extremities of which the forces R and P″ are applied; we shall consequently have, by the property of the lever, when an equilibrium subsists,

$$P'' : R :: MH : MF.$$

Again, the planes of the wheel and of the section EOH being perpendicular to the axis of the cylinder, the triangles HIM MCF are right-angled and similar: hence,

$$MH : MF :: HI : CF.$$

From these proportions we deduce

$$P'' : R :: HI : CF.$$

Let φ represent the angle FPK (*Fig.* 118 and 119), we shall have

$$FPK = DFC = \varphi,$$

and consequently,

$$FK = FP \times \sin\varphi, \quad DC = CF \times \sin\varphi;$$

or,

$$P'' = P\sin\varphi, \quad CF = \frac{DC}{\sin\phi};$$

these values being substituted in the preceding proportion, give

$$P \times \sin\phi : R :: HI : \frac{DC}{\sin\varphi};$$

whence,

$$P \times DC = R \times HI:$$

and from this we deduce the following proportion,

$$P : R :: HI : DC \ldots\ldots (122).$$

It thus appears that the conditions of equilibrium in the wheel and axle require that *the power shall be to the resistance as the radius of the cylinder to that of the wheel.*

243. The pressures sustained by the pivots A and B arise from three distinct causes, viz.: the power, the resistance, and the weight of the machine. If T represent the value of this weight, the centre of gravity of the machine being situated at the point G, we may regard the weight T as suspended from the point G: the machine being symmetrical with respect to its axis, this point will be situated upon the axis. Then, if the power P be replaced by its components

P′ and P″, it will be simply necessary to substitute for the four forces P′, P″, R, and T, two others applied at A and B respectively.

The forces R and T having been determined by experiment, P′ and P″ may be expressed in functions of R. For, we have (*Fig.* 118 and 119)

$$P'=FL=P\cos FPK,\quad P''=FK=P\sin FPK;$$

or,

$$P'=P\cos\varphi,\quad P''=P\sin\varphi\ \ldots\ldots\ (123).$$

But the angle φ being equal to the angle CFD, we obtain

$$1:\cos\varphi::CF:DF,\quad 1:\sin\varphi::CF:CD;$$

whence,

$$\cos\varphi=\frac{DF}{CF},\quad \sin\varphi=\frac{CD}{CF}.$$

Substituting these values in equations(123), there results

$$P'=P\frac{DF}{CF},\quad P''=P\frac{CD}{CF};$$

and replacing P by its value given in the proportion (122), we obtain

$$P'=\frac{R.HI.DF}{DC.CF},\quad P''=\frac{R.HI}{CF}.$$

The vertical forces R and P″ being regarded as acting at the extremities of a lever whose fulcrum is situated at the point M, their resultant will pass through this point, and its value will be expressed by R+P″.

If Z and Z′ denote the effects produced by this resultant upon the points A and B, their values will be determined by the proportions

$$AB:BM::R+P'':Z,$$
$$AB:AM::R+P'':Z'.$$

Representing in like manner by U and U′, the components of T acting on the points of support, we shall have

$$AB:BG::T:U,$$
$$AB:AG::T:U'.$$

The forces U and U′ being vertical, they must be added to Z and Z′ respectively. The horizontal force P′, which acts at C, the centre of the wheel, being likewise decomposed into two components Y and Y′ applied at the points A and B, the values of these components Y and Y′ will result from the proportions

$$AB : CB :: P' : Y,$$
$$AB : AC :: P' : Y'.$$

Thus, having constructed two rectangles, the first of which shall have a height Z+U and a base Y, and the second a height Z′+U′ and a base Y′, the diagonals of these rectangles will represent the pressures on the points of support; and the angles formed by the diagonals with the sides of the rectangles will make known the directions in which these pressures are exerted.

244. If regard be had to the thickness of the cords, we must consider the effects of the powers as transmitted through the axes of the cords; thus, the radius of the cylinder and that of the wheel must be increased by the semi-diameter of the cord, and we shall then have the proportion: *the power is to the resistance as the sum of the radii of the cylinder and cord to the sum of the radii of the wheel and cord.*

245. The *capstan* is a variety of the wheel and axle, in which the axis of the cylinder has a vertical position.

246. Let it now be supposed that we have a system of wheels and axles arranged in the following order:

The power P applied to the circumference of the wheel AD (*Fig.* 120) communicates motion to the cylinder BC, from which the motion is transmitted to a second wheel A′D′, by means of the cord BA′. The wheel A′D′ turns the axle O′B′, to which is attached the cord B′A″, and a similar arrangement is continued to the last axle, from which the resistance R is suspended.

When the system is in equilibrio, if we denote by T, T′, T″, &c., the tensions of the cords BA′, B′A″, &c., we shall have

For the first wheel and axle, $P : T :: OB : OA$,
For the second . . . . $T : T' :: O'B' : O'A'$,
For the third . . . . . . $T' : R :: O''B'' : O''A''$.

These proportions being multiplied together, there results

$$P : R :: OB \times O'B' \times O''B'' : OA \times O'A' \times O''A'';$$

whence,

$$\frac{P}{R} = \frac{OB \times O'B' \times O''B''}{OA \times O'A' \times O''A''},$$

from which we conclude that *the power is to the resistance as the continued product of the radii of the axles to the continued product of the radii of the wheels.*

If the radius of each axle be supposed equal to the n^{th} part of the radius of its wheel, the preceding proportion will become

$$P : R :: \frac{OA}{n} \times \frac{O'A'}{n} \times \frac{O''A''}{n} : OA \times O'A' \times O''A'',$$

which reduces to

$$P : R :: 1 : n^3.$$

247. The different parts of a system of wheel-work are frequently caused to act upon each other by means of teeth projecting from the several circumferences. These teeth perform the same office as the cords in *Fig.* 120. Each toothed-wheel is traversed by an axis bearing a smaller wheel which is called a *pinion*, and the teeth of this pinion are called *leaves*. The first wheel turns its own pinion, both being firmly connected with the same axis, and the leaves of the pinion catching into the teeth of the second wheel, communicate a motion to it in a direction contrary to that of the first wheel. In a similar manner, the pinion of the second wheel transmits a motion to the third wheel, and the same arrangement is continued throughout the system. The pinions replace the axles of the preceding combination, and hence the condition of equilibrium is, that *the power shall be to the resistance as the continued product of the radii of the pinions to the continued product of the radii of the wheels.*

248. Let D, D′, D″, &c. represent the numbers of teeth in the wheels A, A′, A″, &c. (*Fig.* 121), and d, d', d'', &c. the numbers of leaves in the pinions a, a', a'', &c.; and let us suppose that while the wheel A makes N turns, the wheels A′, A″, &c. make respectively N′, N″, &c. turns. At each revolution of the wheel A, the pinion a will engage in succession all its leaves in the teeth of the wheel A′; so that in N revolutions it will engage with A′, a number of teeth expressed by Nd: in like manner, the wheel A′ making N′ turns must engage with the pinion a, a number of teeth expressed by N′D′, and since the numbers of teeth and leaves which the

wheel A′ and the pinion a mutually interlock are equal to each other, we must necessarily have

$$N'D' = Nd;$$

for a similar reason, the other wheels will furnish the equations

$$N''D'' = N'd', \quad N'''D''' = N''d'', \text{ \&c.}$$

These equations being multiplied together, there results

$$N'''D'D''D''' = Ndd'd'';$$

whence,

$$N''' = N\frac{dd'd''}{D'D''D'''}.$$

For example, if it were required to determine the number of teeth which should be employed in order that the wheel A‴ should make one revolution while the wheel A performs 60, we should have

$$N''' = 1, \quad N = 60, \quad 1 = 60\frac{dd'd''}{D'D''D'''} \dots\dots (124).$$

The numbers d, d' and d'' being assumed arbitrarily, we will suppose $d=4$, $d'=5$, $d''=7$; this supposition will reduce the last of the equations (124) to

$$D' \times D'' \times D''' = 60 \times 4 \times 5 \times 7 = 8400.$$

The number 8400 being divided into the three factors 12, 25, and 28, will evidently furnish a solution to the problem, since the quantities D′, D″, and D‴ may be made respectively equal to these factors. The problem obviously admits of an indefinite number of solutions.

The quantity N‴ must be assumed less than N, since we have supposed $d < D'$, $d' < D''$, $d'' < D'''$, and the wheel A‴ will therefore make a less number of revolutions in a given time than the wheel A.

249. The theory of the jack-screw is likewise to be referred to that of the wheel and axle. There are two varieties of this machine, the simple and the compound. The simple jack is composed of a toothed bar of iron AB (*Fig.* 122) which slides in a case CD. The teeth of this bar work in the leaves of the pinion EF, which is put in motion by means of a crank G; thus, the teeth of the bar being subjected to a pressure from the leaves of the pinion, the bar will move in

the direction of its length, and will overcome a resistance at A. In this machine, the crank and pinion perform the offices of the wheel and the axle in the common machine, and the conditions of equilibrium may therefore be stated thus: *the power is to the resistance as the radius of the pinion to the radius of the crank.*

250. In the compound jack-screw, the motion is communicated by means of a crank to a pinion, the leaves of which work into the teeth of a wheel; the axis of this wheel carries a second pinion, which in its turn communicates motion to a second wheel, and the same arrangement is continued to the last pinion, whose leaves act on the teeth of the iron bar.

The condition of equilibrium in this machine obviously is, *that the power shall be to the resistance as the continued product of the radii of the pinions to the continued product of the radii of the wheels and the radius of the crank.*

Of the Inclined Plane.

251. This machine consists of a plane inclined to the horizon: its object is to support in part the weight of a body placed upon it.

Let M represent a body (*Fig.* 123) the weight of which is supposed concentrated at its centre of gravity, and exerted in the vertical direction MP. In order that this body may be sustained in equilibrio upon the inclined plane by the application of a force Q, it is necessary that this force Q, and the weight of the body represented by P, should have a single resultant; this condition can only be fulfilled when the directions of the forces intersect at some point M: but the line MP being vertical, and passing through the centre of gravity, the plane of the forces PMQ must likewise be vertical, and must contain the centre of gravity. Thus the first condition of equilibrium requires that the direction of the resultant be situated in a vertical plane passing through the centre of gravity of the body. The second condition is, that the resultant MN of the two forces P and Q shall be destroyed by the resistance of the inclined plane, which condition can only be satisfied when the direction of this resultant is perpendicular to the plane, and intersects it at some point within the polygon formed by connecting the extreme points of contact of the body and the plane.

252. The preceding conditions being fulfilled, we will suppose KL to represent a body (*Fig.* 123) retained in equilibrio upon an inclined plane by the application of a force Q. Let the lines ME and MF be taken proportional to the weight P and the force Q, and let the parallelogram FMER be constructed: the diagonal MR will represent the pressure exerted by the body against the plane, and if this pressure be denoted by R, we shall have

$$Q : P : R :: \sin PMR : \sin QMR : \sin PMQ \; \ldots\ldots \; (125).$$

The triangles APO and OMN being similar, the angles PMR and CAB will be equal to each other, and therefore

$$\sin PMR = \sin A = \frac{CB}{AC};$$

this value being substituted in the proportion (125), we obtain

$$Q : P : R :: CB : AC \times \sin QMR : AC \times \sin PMQ.$$

253. If the direction of the power be parallel to the plane (*Fig.* 123), the triangles MER and ACB will be similar, since the angles C and E are then equal to each other, and we have the proportion

$$ER : ME :: CB : AC;$$

from which we conclude that when the power acts parallel to the plane, *the power* Q *is to the weight* P *as the height of the plane is to its length.*

254. When the power becomes parallel to the base of the plane (*Fig.* 124), the similar triangles MER and CAB give the proportion

$$ER : EM :: CB : AB,$$

or,

$$Q : P :: CB : AB;$$

thus, in this case, *the power is to the weight as the height of the plane is to the base.*

255. The angle A being supposed equal to 45°, and the power applied parallel to the base, the weight and power will become equal; if the angle A be less than 45°, the weight will

be greater than the power, and if A be greater than 45°, the power will exceed the weight, or the use of the machine will occasion a loss of power.

256. If a body be sustained in equilibrio between two inclined planes, the conditions of equilibrium will require that the weight of the body be susceptible of being resolved into two components which shall be respectively perpendicular to these planes, and shall intersect them at points situated within the polygons formed by joining the points of contact of the body with each plane. The line of direction of the weight being vertical, the plane of its components will likewise be vertical: and since these components are respectively perpendicular to the inclined planes, their plane will be perpendicular to the common intersection of the inclined planes: hence, this intersection must be a horizontal line.

The pressures sustained by these planes may be readily determined by constructing the parallelogram of forces, whose diagonal shall represent the weight of the body, and whose sides shall be perpendicular to the inclined planes.

Of the Screw.

257. Let the sides of the rectangle AM′ (*Fig.* 125) be divided into equal parts by the parallel lines BB′, CC′, &c., and let the diagonals AB′, BC′, &c. be drawn. If the rectangle M′A be then applied to the surface of a right cylinder with a circular base, the circumference of which is equal to the line AA′, in such manner that the right lines MA and M′A′ shall be caused to coincide, the points A, B, &c. will fall upon the points A′, B′, &c. respectively, and the diagonals will trace upon the surface of the cylinder PQNM (*Fig.* 126) a curve PRSTUV &c., which is called a *helix.*

258. The characteristic property of this curve is that the tangent at every point is equally inclined to the element of the cylinder passing through that point: this is obvious from the manner in which the curve is generated.

The distances mn, $m'n'$, $m''n''$, &c. (*Fig.* 125) being equal, their equality will be preserved when the rectangle is applied to the surface of the cylinder: consequently, if we assume *mn* as the base of an isosceles triangle *mno*, the plane of which passes through the axis of the cylinder, and cause the triangle to move around the cylinder, in such manner that the points *m* and *n* shall constantly remain on two adjacent helices, the plane of the triangle continuing to pass through the axis of the cylinder, there will be generated by this motion a projecting fillet which will completely envelop the cylinder MQ. The cylinder and fillet taken conjointly constitute the screw, and the latter is usually called the thread of the screw. This thread is sometimes generated by the motion of a rectangle, instead of a triangle.

259. The nut is composed of a hollow piece, having a spiral groove cut in its interior, in which the threads of the screw work. It may be regarded as forming the mould of a portion of the screw.

The screw can be readily turned within the nut, and at each revolution passes over a distance in the direction of its length equal to the distance between the threads.

Since the conditions of the problem are precisely the same, whether we regard the nut as turning on the screw, or the screw as turning within the nut, we will adopt the first hypothesis.

260. To determine the conditions of equilibrium in this machine, we will suppose the nut to be placed on its screw, and the axis of the screw to have a vertical position. Let the nut be divided into any number of particles, whose weights are denoted by m, m', m'', &c., each of which rests on some point of the screw; and let us determine the force necessary to sustain any one particle m (*Fig.* 127).

The particle m, being connected with the axis of the screw in such manner that its distance from the axis shall remain invariable, must, if unsupported, descend along a helix, every point of which will be at the distance mC from the axis. Thus, by regarding this helix as an inclined plane, the height of this plane will be the distance between the threads, and its base will be the circumference described with mC as a radius.

Let us suppose a horizontal force P (*Fig.* 128) to be applied immediately to the particle m, for the purpose of sustaining it in equilibrio upon the inclined plane. By constructing the right-angled triangle KHm, whose height shall be the distance between the threads, and its base the circumference described with the radius mC, we shall obtain by the principle of the inclined plane (Art. 254),

$$P : m :: \text{height} : KH;$$

or,

$$P : m :: mH : \text{circumference } Cm \ldots\ldots (126).$$

But if the point of application of the power be transferred from the point m to the point D, the extremity of the lever CD, the force Q, which applied at this point will produce the same effect as the force P applied at m, can be determined from the following proportion,

$$Q : P :: Cm : CD;$$

or,

$$Q : P :: \text{circumference } Cm : \text{circumference } CD.$$

And by comparing this proportion with (126), we obtain

$$Q : m :: mH : \text{circumference } CD.$$

Thus, for the particle m, the power is to the weight as the distance between the threads is to the circumference described by the power.

This proportion being true, whatever may be the distance of the particle m from the axis of the cylinder, we shall obtain for the other points in the surface of the screw, which support the weights m', m'', &c., by means of the forces Q', Q'', &c., applied at the same distance CD,

$$Q' : m' :: mH : \text{circumference } CD,$$
$$Q'' : m'' :: mH : \text{circumference } CD,$$
$$Q''' : m''' :: mH : \text{circumference } CD,$$

&c. &c. &c.

From these proportions and the preceding, we deduce

$$Q = \frac{m \times mH}{\text{circumf. } CD},\ Q' = \frac{m' \times mH}{\text{circumf. } CD},\ Q'' = \frac{m'' \times mH}{\text{circumf. } CD}. \ . \ (127).$$

These values are independent of the distances of the points m, m', m'', &c., from the axis of the cylinder; and since the forces Q, Q', Q'', &c, were supposed applied at equal distances from the axis, they will communicate to the nut the same motion of rotation as would be imparted by a single force equal to their sum, and acting along the line DQ. Thus, by adding the equations (127), we find

$$(m+m'+m''+\&c.)=(Q+Q'+Q''+\&c.)\frac{\text{circumf. CD}}{mH};$$

and since the sum $(m+m'+m''+\&c.)$ represents the entire weight M of the nut, we shall have, after replacing the sum of the forces Q, Q', Q'', &c., by a single force $Q_{\prime}$,

$$M=Q_{\prime}\times\frac{\text{circumf. CD}}{mH}:$$

whence,

$$Q_{\prime} : M :: mH : \text{circumference CD}:$$

or, *the power is to the weight as the distance between the threads is to the circumference described by the power.*

It thus appears that the machine will be rendered more powerful by applying the force at a greater distance from the axis, or by diminishing the distance between the threads of the screw.

Of the Wedge.

261. The wedge is a triangular prism, one of whose edges is introduced into the crevice of a body, for the purpose of enlarging the opening.

All cutting instruments, such as knives, scissors, razors, &c., may be regarded as wedges.

262. The power is usually applied by communicating an impulse to the back of the wedge, in a direction perpendicular to it: if the direction of this impulse be oblique, it may always be resolved into two components, of which one shall be perpendicular to the back of the wedge, and the other shall coincide with it. The first will produce its entire effect, the second will only tend to move the point of application of the power along the back of the wedge.

Let ABC (*Fig.* 129) represent a profile of the wedge; AC and BC are sections of its faces, and AB a section of its back, upon which the power is applied in a perpendicular direction.

To determine the relation between the power applied to the back of the wedge and the pressures exerted at the faces, we will suppose the power F to be represented by the line DE, and draw DM and DN perpendicular to the faces AC and BC: then, by constructing the parallelogram DIEK, the components DI and DK will represent the pressures exerted against AC and BC. Denoting these pressures by X and Y, the similar triangles ABC and IDE give the proportion

$$DE : DI : IE :: AB : AC : BC;$$

or,

$$F : X : Y :: AB : AC : BC;$$

and by multiplying the three last terms in this proportion by the line GH (*Fig.* 130), we have

$$F : X : Y :: AB \times GH : AC \times GH : BC \times GH.$$

The products AB × GH, AC × GH, and BC × GH express the surfaces of the back and faces of the wedge, and we therefore conclude that in this machine, *the power* F *applied to the back, and the efforts* X *and* Y *exerted by the sides, are respectively proportional to the surfaces of the back and sides of the wedge.*

The power of the wedge will evidently be augmented either by decreasing the back of the wedge, or by increasing the lengths of its faces.

Friction.

263. If a body be placed upon a horizontal plane, the action of gravity exerted upon it will be entirely counteracted by the resistance of the plane, and the least possible impulse will communicate a motion to the body, if it be not retained by physical causes which oppose motion. The most efficient of these causes is friction. This term is applied to the force which tends to prevent a body from sliding along the surface of a second body, and which arises from the slight inequalities in the two surfaces; the projecting points of one surface entering the cavities of the second give rise to a passive force which tends to assist or oppose the power, according as this power is employed to sustain or move the body.

The effect of friction is found to be sensibly proportional to the pressure, so long as this pressure is retained within moderate limits. Thus, if we denote by f the friction exerted by a homogeneous body AB (*Fig.* 131), the weight of which is equal to unity, and if AB′ be supposed equal to twice AB, the corresponding friction will be expressed by $2f$; if AB″ be triple AB, the friction will be equal to $3f$, &c.; so that if F denote the friction exerted by the body AM, which contains a number N of units of weight, we shall have

$$F=Nf \ldots\ldots (128).$$

264. The friction may be measured in the following manner:

Let AB (*Fig.* 132) represent the body which exerts by its weight the unit of pressure on a horizontal plane LK. To the body is attached a thread CDE, which passes over a fixed pulley, and sustains the weight M: this weight being gradually increased, its intensity at the moment when it is about to overcome the resistance which the body opposes to motion, will measure the friction f, corresponding to the unit of pressure.

265. There is another method of measuring the friction, which results from the following theorem: *If a body* MN *be placed upon an inclined plane* AC (*Fig.* 133), *and if the angle* A *which this plane forms with the horizon be gradually augmented until the body is about to commence sliding upon the plane, the numerical value of the unit of friction will then be equal to the tangent of the angle which the inclined plane forms with the horizon.*

To demonstrate this fact, let the lines GD and GK be drawn, respectively perpendicular to AB and AC; the centre of gravity of the body being supposed situated at the point G. Represent by GD the weight of the body, and decompose GD into two forces GH and GK, parallel and perpendicular to the inclined plane: we shall then have

$$GH=DK=GD \sin DGK,$$
$$GK=GD \cos DGK;$$

but the angles DGK and CAB are equal to each other; and hence, the preceding equations may be written thus,

$$GH = GD \times \sin A,$$
$$GK = GD \times \cos A;$$

or if N expresses the weight of the body,

$$GH = N \sin A,$$
$$GK = N \cos A.$$

The pressure sustained by the inclined plane being expressed by $GK = N \cos A$, the corresponding friction will be expressed by $N \cos A . f$; but since the effect of friction is to counteract that tendency which the body has to move along the plane when there is no friction, it follows, that an equilibrium will subsist between the force of friction and the component of the force of gravity, $GH = N \sin A$, which acts in the direction of the plane; whence we obtain

$$N \cos A . f = N \sin A.$$

From this equation we deduce

$$f = \text{tang}\, A \, \ldots\ldots \, (129).$$

266. The angle thus determined is called the angle of friction; its value will remain constant only when we adopt the hypothesis that the friction varies proportionally to the pressure. For, the relation expressed in (129), has been deduced by employing (128), which expresses this law; and the law, as has been already remarked, exists only for moderate pressures.

267. Since different substances have pores of very unequal magnitudes it happens that the friction is not the same for all bodies; hence, experiments have been instituted for the purpose of determining the friction peculiar to each.

The following results which express the relation between the friction and the pressure, have been obtained by *Coulomb:*

Iron against iron	$f = 0.28$,
Iron against brass	$f = 0.26$,
Oak against oak	$f = 0.43$,
Oak against fir	$f = 0.65$,
Fir against fir	$f = 0.56$,
Elm against elm	$f = 0.47$.

These last results were obtained when the friction was exerted in the direction of the fibres; but when the direction of the fibres formed a right angle with that of the motion, the friction was found to be much less, but still in a constant ratio to the pressure; the results in this case were as follows:

Oak against fir . . . . . . $f=0.158$,
Fir against fir . . . . . . $f=0.167$,
Elm against elm . . . . . $f=0.100$.

It also appears from the experiments of Coulomb, that the friction exerted by a body in motion is very nearly independent of the velocity of the body.

The polish of the body and the introduction of an unctuous substance between the rubbing surfaces contribute to lessen the effect of the friction.

268. When one body is caused to roll upon another, a certain degree of resistance is still offered by friction, but this resistance is much less intense than in the case of a sliding motion. This result appears to be a consequence of the disengagement of the inequalities in the surfaces, which the motion of rotation tends to effect.

269. The general laws of friction, as deduced from the experiments of Coulomb, may be summed up as follows:

1°. *Friction varies with the polish of the surface:* Thus, the resistance opposed by friction may be reduced by diminishing the asperities of the rubbing surfaces.

2°. *The friction between bodies of the same kind is greater than between bodies of different kinds.*

3°. *Friction does not depend on the extent of surface in contact, the entire pressure exerted between the bodies remaining the same.*

4°. *Friction is proportional to the pressure.*

5°. *Friction is diminished by interposing a substance of an unctuous nature between two surfaces which slide upon each other.*

6°. *The friction is greatly diminished by substituting a rolling for a sliding motion.*

270. The adhesion which takes place between the surfaces of bodies is another physical cause opposed to their motion.

It is difficult to estimate, in a precise manner, the proper measure of this effect, in consequence of its being liable to a very great increase with time in those machines which are at rest; and, on the contrary, to undergo occasional changes in those which are in motion.

The law which this force usually follows is that of being sensibly proportional to the extent of the adhering surfaces. Thus, by denoting the adhesion of a superficial unit by the quantity ψ, the adhesion of a surface whose area is a will be expressed by $a\psi$.

Effects of Friction in certain Machines.

271. Let P and S (*Fig.* 134) represent two forces applied to a material point which rests in equilibrio on an inclined plane AB, and let α and α' denote the angles which the directions of these forces make with the plane. If we disregard the effects of friction and adhesion, the conditions of equilibrium will require the relation

$$P \cos \alpha = S \cos \alpha' \; . \; . \; . \; . \; . \; (130);$$

but if friction and adhesion be considered, since these two forces are opposed to the motion which the power P tends to impress in a direction from m towards B, it will be necessary to add these forces to the component of S in the direction of the plane, which is expressed by $S \cos \alpha'$. To determine their values, we remark that the pressure exerted upon the inclined plane is produced by the normal components of the forces P and S. These components are expressed respectively by $P \sin \alpha$ and $S \sin \alpha'$; and their sum will be equal to the entire pressure which is denoted by N in equation (128). Thus, the force arising from friction is expressed by $(P \sin \alpha + S \sin \alpha')f$. If we denote by a the area of the surface in contact with the plane, the adhesion will be represented, as has been before stated, by the quantity $a\psi$. Consequently, by adding these forces to the second member of the equation (130), we shall obtain for the condition of equilibrium

$$P \cos \alpha = S \cos \alpha' + S \sin \alpha' f + P \sin \alpha f + a\psi;$$

from which we deduce

$$P=\frac{S\cos\alpha'+Sf\sin\alpha'+a\psi}{\cos\alpha-f\sin\alpha}\;.....\;(131).$$

272. If, on the contrary, the power be only required to retain in equilibrio the point m, the friction and adhesion, being still opposed to motion, will tend to assist the force P, and the algebraic signs of these quantities must therefore be changed. Representing by P′ the force necessary to support m, upon this hypothesis, we shall have

$$P'=\frac{S\cos\alpha'-Sf\sin\alpha'-a\psi}{\cos\alpha+f\sin\alpha}\;.....\;(132).$$

It is evident that the equilibrium may be preserved by the application of any force P″ in the direction Pm, provided the intensity of this force be intermediate between the intensities P and P′ given by equations (131) and (132).

273. The effect of friction in modifying the conditions of equilibrium in the lever and pulley will now be considered.

Let the lever be perforated by a circular hole, through which is passed a cylinder having a vertical position. Since the circumstances will be the same, whether we regard the lever as turning about the cylinder, or the cylinder as turning within the lever, we shall adopt the first hypothesis, and consider the point m of the lever (*Fig.* 135), which, being in contact with the cylinder, is subjected to the action of the force of friction. Let the cylinder be intersected by a horizontal plane passing through m, and let this plane be assumed as the co-ordinate plane of x, y. For the purpose of simplifying the question, we shall suppose the resultant R of all the forces applied to the lever to be situated in the plane of x, y.

The intersections of the cylinder and lever by the plane of x, y will be represented respectively by the circle mBE, and the plane curve GIL. The cylinder being immoveable, the point m can be subject only to a circular motion about the point C, at which the axis of the cylinder is intersected by the plane of x, y. If the point m remain immoveable, the equilibrium must result from the combined actions of the resultant R of the several forces applied to the lever, the friction, and the resistance opposed by the axis. The direction of this resistance being normal to the surface of the cylinder, we may drop the consideration of the fixed cylinder, and consider the point as perfectly free, and sustained in equilibrio by the three following forces: 1°. the normal force, which acts in the direction from C towards m; 2°. the friction, which acts along the tangent mD; 3°. the resultant R of all the forces in the system.

274. It should be remarked that although two of the three forces are applied at m, the third force may be applied at any other point, provided its line of direction passes through m.

If we regard the point of application of the third force as unknown, the conditions of equilibrium of the three forces will be expressed by the equations (52), (53), and (54).

275. To express these conditions, we will suppose the origin of co-ordinates to be placed at C, and represent by N the normal force, which forms with the axes angles equal to α and β: denote by F the friction, the direction of which forms with the axes the angles α' and β', and by h the radius of the cylinder which is supposed to be nearly of the same size as the circular hole through which it passes. The components of the force R, parallel to the two axes, will be represented by X and Y respectively, and the perpendicular distance of this force from the point C by the letter r.

This being premised, the condition expressed by equation (52) requires that the sum of the components parallel to the axis of x shall be equal to zero; hence,

$$\mathrm{N}\cos\alpha+\mathrm{X}+\mathrm{F}\cos\alpha'=0\ .\ .\ .\ .\ .\ (133).$$

For a similar reason, the components parallel to the axis of y give

$$\mathrm{N}\cos\beta+\mathrm{Y}+\mathrm{F}\cos\beta'=0\ .\ .\ .\ .\ .\ (134).$$

And the third equation of equilibrium, which expresses the relation between the moments, gives

$$\mathrm{R}r+\mathrm{F}h=0\ .\ .\ .\ .\ .\ (135);$$

which becomes, by substituting for F its value deduced from equation (128),

$$\mathrm{R}r+\mathrm{N}fh=0\ .\ .\ .\ .\ .\ (136);$$

276. Before employing equations (133) and (134), it may be remarked that any one of the four quantities cos α, cos α', cos β, cos β', which appear in those expressions, will serve to

determine the remaining three. For, the angle yCx (*Fig.* 135) being equal to a right angle, we shall have

$$\cos\beta=\sin\alpha;$$

and if we draw the line FK parallel to Cm (*Fig.* 136), we shall obtain

$$m\text{FH}=m\text{FK}+\text{KFH};$$

or,

$$\alpha'=90^\circ+\alpha;$$

consequently,

$$\cos\alpha'=\cos 90^\circ\cos\alpha-\sin 90^\circ\sin\alpha=-\sin\alpha,$$

$$\cos\beta'=\sin\alpha'=\sin 90^\circ\cos\alpha+\cos 90^\circ\sin\alpha=\cos\alpha.$$

By means of these values of $\cos\beta$, $\cos\alpha'$, and $\cos\beta'$, we reduce the equations (133) and (134) to

$$\left.\begin{array}{l}\text{N}\cos\alpha+\text{X}-\text{F}\sin\alpha=0\\ \text{N}\sin\alpha+\text{Y}+\text{F}\cos\alpha=0\end{array}\right\}\ldots\ldots(137).$$

277. These equations admit of a further reduction, from the consideration that the friction exerted at the point m is proportional to the normal pressure N; thus, by replacing F by its value Nf in the equations (137), we find

$$\left.\begin{array}{r}\text{X}=\text{N}f\sin\alpha-\text{N}\cos\alpha\\ \text{Y}=-\text{N}f\cos\alpha-\text{N}\sin\alpha\end{array}\right\}\ldots\ldots(138).$$

But X and Y being rectangular components of the force R, we must have the relation

$$\text{R}^2=\text{X}^2+\text{Y}^2.$$

Substituting in this equation the values of X and Y found above, we obtain

$$\text{R}^2=\text{N}^2(\sin^2\alpha+\cos^2\alpha)+\text{N}^2f^2(\sin^2\alpha+\cos^2\alpha),$$

or,

$$\text{R}^2=\text{N}^2(1+f^2)\ldots\ldots(139).$$

From this equation taken in connexion with (136), we find

$$r=\pm\frac{fh}{\sqrt{(1+f^2)}}\ldots\ldots(140).$$

This value of r will always be less than that of h, since the fraction $\dfrac{f}{\sqrt{(1+f^2)}}$ is less than unity; but h represents the radius of the cylinder, and hence it follows that the equilibrium is only possible when the distance r of the point C from the direction of the resultant does not exceed the radius of the cylinder. The direction of the resultant will therefore intersect the surface of the cylinder. This condition, without which the equilibrium of the lever, maintained by the effect of friction, becomes impossible, is not alone sufficient; for the value of r must not exceed that determined by equation (140); otherwise the condition of moments could not be fulfilled.

278. It may be remarked, that the equation of the moments expresses the condition that the friction and the resultant of all the forces applied to the lever, acting conjointly, will prevent any tendency to rotation. For since the direction of the normal force passes through the origin, it can have no tendency to produce rotation. If, therefore, an equilibrium subsists, it must be produced in consequence of the forces R and F exerting equal efforts to turn the system in contrary directions. But this is precisely the condition expressed by the equation (135), since the moments of the forces are equal and have contrary signs.

279. We can also determine the relation which must subsist between the power and the resistance. For this purpose the preceding results must undergo certain modifications.

Let P and S represent the power and resistance (*Fig.* 137), which form with each other an angle θ; the resultant of these two forces will be determined by the equation (Art. 30)

$$R^2=P^2+S^2+2PS\cos\theta.$$

By substituting this value of R^2 in equation (139), it becomes

$$P^2+S^2+2PS\cos\theta=N^2(1+f^2)\ \ldots\ldots\ (141).$$

280. Let the value of N be now expressed in functions of the quantities P and S. For this purpose, let the perpendiculars p and s be demitted on the directions of the forces P and S respectively; the moment Rr of the resultant can then be changed into $Pp-Ss$, or $Ss-Pp$, according to the direction in which the resultant tends to turn the system; thus the equation (136) will become

$$\pm(Pp-Ss)+Nfh=0$$

whence,

$$N^2=\frac{(Pp-Ss)^2}{f^2h^2};$$

and by substituting this value in equation (141), we find

$$P^2+2PS\cos\theta+S^2=\frac{(Pp-Ss)^2}{h^2}\times\frac{1+f^2}{f^2}.$$

This result may be simplified by making

$$P=Sz,\quad\text{and}\ \frac{1+f^2}{f^2}=k^2\ \ldots\ldots\ (142).$$

The quantity S^2 will then disappear, being a common factor, and the equation will reduce to

$$z^2+2z\cos\theta+1=\frac{k^2}{h^2}(pz-s)^2.$$

From this equation we deduce

$$z^2h^2+2zh^2\cos\theta+h^2=k^2(p^2z^2-2pzs+s^2);$$

and by transposition,

$$(k^2p^2-h^2)z^2-2(psk^2+h^2\cos\theta)z+k^2s^2-h^2=0;$$

or, by division,

$$z^2-\frac{2(psk^2+h^2\cos\theta)}{k^2p^2-h^2}z+\frac{k^2s^2-h^2}{k^2p^2-h^2}=0.$$

The value of z deduced from this equation is the ratio of the power to the resistance; and since z has two values, it is obvious that the first will apply to the case in which the power is about to overcome the resistance, and the second to that in which the resistance is about to overcome the power. By resolving the equation, we find

$$z=\frac{psk^2+h^2\cos\theta\pm\sqrt{[(psk^2+h^2\cos\theta)^2-(k^2p^2-h^2)(k^2s^2-h^2)]}}{k^2p^2-h^2};$$

and by developing and reducing the terms contained under the radical signs, we obtain

$$z=\frac{psk^2+h^2\cos\theta\pm h\sqrt{[k^2(p^2+2ps\cos\theta+s^2)-h^2(1-\cos^2\theta)]}}{k^2p^2-h^2};$$

and finally, by substituting for z and $1-\cos^2\theta$ their respective values, we shall have

$$\frac{P}{S}=\frac{psk^2+h^2\cos\theta\pm k\sqrt{[k^2(p^2+2ps\cos\theta+s^2)-h^2\sin^2\theta]}}{k^2p^2-h^2}.$$

281. If the radius of the cylinder be very small, its square h^2 may be neglected, and the preceding ratio will then become

$$\frac{P}{S}=\frac{s}{p}\pm\frac{h\sqrt{(p^2+2ps\cos\theta+s^2)}}{kp^2}.$$

If the perpendiculars p and s, demitted from the point C on the respective directions of the power and resistance, become equal to each other, the results will apply to the case of the pulley; and by still neglecting the quantity h^2, we shall find

$$\frac{P}{S}=1\pm\frac{h\sqrt{[2(1+\cos\theta)]}}{kp}\ \ldots\ldots\ (143).$$

282. Finally, when the power and resistance act in parallel directions, the angle θ becomes equal to zero; whence,

$$\sin\theta=0,\quad \cos\theta=1;$$

and the equation (143) then reduces to

$$\frac{P}{S}=1\pm\frac{2h}{kp}.$$

283. The same principles will serve to determine the conditions of equilibrium in the other mechanical powers, when regard is had to the effects of friction; but the results obtained would in general prove much more complicated.

Of the Stiffness of Cordage.

284. In employing the cord as a means of transmitting the effect of a force to a machine, we have hitherto supposed the cord to be perfectly flexible. But as this hypothesis is inadmissible in practice, it becomes necessary to estimate the additional force that will be necessary to overcome the rigidity of the cord.

Let P and Q (*Fig.* 138) represent two weights which are applied to the extremities of a cord passing over a fixed pulley: if the weight P be supposed to prevail, and the cord be regarded as perfectly rigid, the extremity Q will evidently be brought into a position Q′, such that the vertical line Q′O will intersect the horizontal line CO drawn through C, at a distance CO from the centre, greater than the radius CG. The extremity P will at the same time assume the position P′, such that the vertical line drawn through P' will intersect the radius CF. Hence the arm of the lever to which the force Q is applied will now be longer than that of the force P, and the condition of equilibrium will therefore require that the force P shall exceed Q.

285. If the cord be supposed imperfectly rigid, similar effects will be produced, though in a less degree; and in practice, it is found that the decrease in the arm of lever, to which the preponderating weight is applied, is wholly insensible. Hence, in estimating the effects produced by the rigidity of a cord employed in a machine, it will simply be necessary to increase the arm of the lever to which the resistance Q is applied, by a proper quantity q.

286. To determine the value of q, we remark that the resistance to flexure opposed by a given cord arises from two distinct causes,—viz. 1°. The tension of the cord, or the force Q which is employed to stretch it; and, 2°. The materials used in the construction of the cord, and the degree of twist which has been given to it. The resistance arising from the tension of the cord is found to be proportional to this tension, and may therefore be represented by an expression of the form bQ, in which b represents an indeterminate constant. The resistance produced by the second cause may be represented by a quantity a.

Thus, for the same cord bent over the same pulley, the expression $(a+bQ)$ may be supposed to represent the effort necessary to bend it. But if we suppose the diameter of a second cord to be greater, the force necessary to bend it will become greater, and we can assume that this force will increase according to some power n of the diameter D. The force will also increase as the curvature increases, or as the radius of the pulley is decreased, and hence $\frac{D^n}{r}(a+bQ)$ may be taken as an expression for the force necessary to overcome the rigidity of the cord. This expression represents the increment that must be given to the power P, in order that it may be on the point of overcoming the resistance Q: but we also have

$$Pr = Q(r+q);$$

and since the forces P and Q become equal when the cord is supposed destitute of rigidity, P—Q or $Q\frac{q}{r}$ will also express the value of this increment. By making these values equal to each other, we obtain

$$D^n(a+bQ)=Qq;$$

whence,

$$q=\frac{D^n}{Q}(a+bQ)\,.\,.\,.\,.\,.\,(143\,a).$$

287. This equation should only be regarded as furnishing an approximate value of the quantity q, since the above relation has been obtained by considerations of a very general character. It moreover contains certain unknown quantities a, b, and n, which vary with different cords.

For the purpose of verifying the truth of the preceding formula, and at the same time determining the values of the unknown constants, we proceed as follows.

Having selected a cord, we pass it over a fixed pulley, and attach to its extremities two equal weights: we then increase one of these weights until it is about to prevail over the other, and the difference k will give one value of the quantity $\frac{D^n}{r}(a+bQ)$.

By repeating the experiment several times, changing the weights, the cord, or the pulley, we can obtain a number of similar equations, in which the quantities a, b, and n will be the same, and the quantities D, r, and Q, although different, will be known by observation. Three such equations will serve to determine a, b, and n, and their values being substituted in the general relation expressed by formula (143 a), the accuracy of the formula can be tested by comparing it with the results furnished by other experiments.

The quantity n was found by *Coulomb* to be usually about 1.7 or 1.8; and the resistance to flexure must therefore vary nearly as the square of the diameter of the cord: but the quantity n is itself subject to some variation, becoming nearly 1.4 when the cord has been long used.

The following results, expressed in French pounds, were obtained in the experiments of Coulomb.

			lbs.		lbs.
White rope	30 threads in a yarn	$\frac{D^n}{r}\times a=$	4.2	$\frac{D^n}{r}b\times 100=$	9
	15 threads	" " "	1.2	"	5.1
	6 threads	" " "	0.2	"	2.2
Tarred rope	30 threads in a yarn	$\frac{D^n}{r}\times a=$	6.6	$\frac{D^n}{r}b\times 100=$	11.6
	15 threads	" " "	2.0	"	5.6
	6 threads	" " "	0.4	"	2.4

On the Resistance of Solids.

288. The particles of every solid body are found to oppose a certain resistance to any force which tends to separate them. This resistance arises from the mutual actions exerted by the particles upon each other; and if the nature of these actions, as well as the arrangement of the particles which compose the body, were accurately known, it might be possible to estimate the force necessary to separate the particles, or to produce a given change in the figure of the body. But as we are entirely ignorant of these particulars, it becomes necessary to adopt some hypothesis relative to the manner in which bodies are constituted, and the nature of the actions exerted by the particles upon each other. Then, by reasoning upon such hypothesis, we can obtain results which, compared with those derived from experiment, will serve to test the accuracy of the supposition.

289. The hypotheses most generally adopted are—1°. That of *Galileo*, which supposes all solid bodies to be made up of fibres, disposed parallel to the length of the body, and susceptible of being ruptured without undergoing flexure, extension, or compression; or, 2°. That of *Leibnitz*, modified by *Bernoulli* and others, which regards the fibres of all bodies as elastic; being susceptible of extension and compression, and capable of opposing a resistance directly proportional to their extensions or compressions. The force required to produce a given extension is, moreover, supposed to be equal to that which is capable of producing an equal compression.

290. It is very certain that neither of these hypotheses is strictly correct; but as the results given by the latter differ but

little from the truth, when the extensions or compressions are inconsiderable, we shall adopt it, and apply it to the investigation of the resistance which a solid will oppose under different circumstances.

291. The kind of resistance which the body offers will depend in a great measure upon the manner in which the force is applied. Thus, the force may exert an effort to extend or compress the solid in the direction of its length, or it may tend to produce a flexure of the solid, or it may operate as a force of torsion; and in each of these cases it may be required to determine the force necessary to produce a rupture or separation of the particles, or simply that necessary to effect a given change in the figure of the solid.

The cases which more generally occur are, 1°. That in which the solid sustains an extension or compression in the direction of its length, without undergoing sensible flexure; and, 2°. That in which flexure is produced by the application of a force perpendicular to the length of the solid.

As it is the object of the present article merely to exhibit the general methods in which the hypothesis assumed may be applied to the determination of the *strength* of bodies, or the resistance which they are capable of opposing, we shall confine our investigations to the consideration of these two cases.

292. The resistance of a body to a change of figure depends upon its force of *elasticity*, which is measured by the effort necessary to compress or extend the body by a given quantity. Its resistance to rupture depends upon its force of *tenacity*, or upon the effort necessary to rupture or crush the body.

The values of these forces having been determined experimentally for a body composed of a given substance, and having a simple form, we can calculate the compression, extension, or flexure produced in another body, of the same substance, by the application of a given force. The methods of effecting this calculation will now be explained.

Of the Resistance to Compression or Extension.

293. When a solid is stretched or compressed in the direction of its length, being at the same time prevented from experiencing flexure, the lengths of its fibres are found to undergo very slight variations, and we can therefore assume, in conformity with the hypothesis adopted, 1°. That the extensions or compressions of all the fibres will be equal to each other, and uniform throughout the extent of each fibre; and that the force necessary to produce a given extension will be capable of producing an equal compression. 2°. That the variations in the lengths, and the resistances opposed by the fibres, are constantly proportional to the forces which produce them; and that this proportion obtains even for those forces which rupture or crush the body.

294. Let a cubical mass of any substance be placed upon a horizontal plane, and subjected to the action of a weight which rests upon its upper surface, compressing the substance in the vertical direction. Denote by

a, the length of one of the edges of the cube;

a', the quantity by which its vertical dimension is compressed, and which is always extremely small in comparison with a;

P, the force which produces the compression.

Then, since the compression of each fibre is supposed uniform throughout, or since the particles which compose any one fibre are supposed to approach each other equally at every point of such fibre; it is obvious that the entire compression a', sustained by any fibre, will be directly proportional to its length a. For example, if the length of another solid be supposed equal to $2a$, its transverse section remaining the same, and if the same force P be applied to its upper surface, the number of particles in the length $2a$ will be twice as great as the number contained in a; and each pair of consecutive particles being caused to approach each other to within the same distance, in order that the resistance of the fibre may be uniform throughout, the whole variation in the length $2a$ will evidently be twice as great as that which was produced in the length a, and will therefore be expressed by $2a'$. And, generally, the compression of the solid, whose length is na, and whose transverse section remains the same, will be expressed by na', when the same force P is applied to its upper surface. Let the quantity a be supposed equal to the linear unit,—one foot, for example; then n will express the number of feet contained in the length of the second solid, and na' will express the variation produced in the length of a solid whose transverse section is equal to one square foot, and whose length is equal to n feet.

295. The preceding remarks have been confined to the case in which the solid suffers compression, but from the nature of the hypothesis, they must apply with equal force to the case in which the effort is exerted to extend the body.

296. If the transverse section of a second solid, whose length is likewise equal to n, be supposed greater than that of the first, the number of its fibres will be increased in the same proportion, and the total effort exerted by these fibres when compressed to the same degree will evidently be proportional to their number: thus, if P′ represent the force necessary to compress a prism whose length is n, and whose transverse section contains m square feet, by a quantity equal to na, we shall have the proportion

$$\text{section } 1 : \text{section } m :: P : P';$$

whence,

$$P' = mP.$$

297. If the force P′ be increased, the solid will undergo a greater compression, and the quantity by which the length n of the fibre is compressed will no longer be represented by na', but by an unknown quantity na''. To determine this quantity, we recur to the hypothesis which assumes that the compressions are proportional to the forces which produce them; hence, by calling P″ the value of the force which produces the compression na'', we shall have

$$na' : na'' :: P' : P'',$$

and therefore,

$$P'' = P' \frac{na''}{na'};$$

or, replacing P′ by its value mP, we have

$$P'' = \frac{P}{a'} \times \frac{m . na''}{n} \; \ldots\ldots \; (143\,b).$$

298. The quantity $\frac{P}{a'}$ is called the *coefficient of the elasticity:* its value will depend only on the elastic force of the substance of which the prism is composed, and will therefore be independent of the dimensions of the particular prism under consideration. If we denote this coefficient by A, we shall obtain, for the entire compression of the prism,

$$na'' = \frac{nP''}{mA}.$$

This expression will determine the quantity by which a given prism will be compressed under the influence of a given force, when the coefficient of the elasticity has been previously ascertained. It should be remembered, however, that this formula is only applicable when the compressions are exceedingly small; and that the solid is ruptured or crushed before its length undergoes a very sensible change.

299. The preceding expression is equally applicable when the force P″ tends to stretch the solid.

300. To determine the force necessary to rupture a given prism, when exerted in the direction of the length of the prism, we shall denote by B the force necessary to rupture a prism of the given substance whose transverse section is a square foot. Then, if the transverse section of the given prism be supposed to contain m square feet, the number of its fibres will be m times greater than the number contained in the prism whose section is equal to one square foot; and since each fibre in the two prisms must oppose the same resistance at the instant of rupture, we shall determine the force P″ necessary to rupture the given prism, by the proportion

$$\text{section } 1 : \text{section } m :: B : P'';$$

whence,

$$P'' = mB.$$

301. The quantity B is called the *coefficient of the tenacity,* and depends only on the nature of the substance under consid-

eration. Having determined this quantity by experiment, we can readily calculate the force necessary to rupture a given prism of the same substance. This investigation is equally applicable whether the force be exerted to compress or extend the solid. The methods of determining experimentally the coefficients of the elasticity and tenacity will be explained hereafter.

Of the Resistance of a Solid to Flexure and Fracture produced by a Force acting at right angles to the direction of the Fibres.

302. When the length of a solid body bears a certain proportion to its thickness, the body is found to undergo a certain degree of flexure before breaking. This flexure becomes more perceptible as the length of the solid is increased: thus a bar of wrought iron whose length does not exceed twelve or fifteen times its thickness gives very slight indications of flexibility; but when its length is increased to forty or fifty times its thickness, it yields readily to an effort exerted to bend it, and becomes susceptible of taking a very considerable flexure before breaking.

303. If a force P be applied in a direction perpendicular to the length of the solid AB (*Fig.* 139), which is supported at its two extremities, and if this force be supposed to produce a certain degree of flexure in the solid, causing it to assume the form represented in *Fig.* 139 *a*, the fibres *aa*, &c. situated on the convex side will be extended, their lengths being increased, and those situated on the concave side will suffer a compression, and will undergo a diminution in length. This effect is readily observed: for, if the force P be gradually increased until it become capable of breaking the solid, the rupture will be found to commence at a point D on the convex side, thereby indicating that the fibres *aa* on that side have been most extended; and if some of the fibres situated on the convex side be previously separated by cutting them through transversely, it will be found that a smaller force than P will be required to fracture the solid. But if, on the contrary, the fibres *bb* situated near the opposite side of the

solid be cut transversely to a certain depth EF (*Fig.* 139), and if a thin plate of some unyielding substance be introduced into the cut EF, so as to fill it entirely, it will be found, upon subjecting the solid to the action of the force P, that the thin plate will be retained by a strong pressure tending to compress it, and that the strength of the solid will not be diminished, the rupture commencing at the convex side, when the force P has been increased in the same degree as was necessary to rupture the solid before severing any of its fibres.

As we proceed from the convex towards the concave side of the solid, the extensions of the fibres will gradually diminish, and at a certain distance from the surface, their lengths will undergo no variation; beyond this distance the extensions will be changed into compressions, and these will again increase until we arrive at the concave side.

304. The flexure of the fibres being supposed to take place entirely in planes parallel to the axis of the solid and the direction of the force applied, it is evident that the change of figure experienced by the solid will require that those fibres whose lengths undergo no variation should be contained, previous to the flexure, in a plane perpendicular to the direction of the force which produces the flexure; and that, after the flexure, these fibres will form a cylindrical surface, whose elements will be parallel to the same plane. Moreover, the fibres situated at equal distances from this plane will undergo equal extensions or compressions.

305. Let us now conceive a right prism AB to be firmly fixed at its extremity A, in such manner that its axis shall be horizontal, and that a vertical plane passing through the axis shall divide the solid into two symmetrical parts. Let a weight P be applied at the other extremity of the solid, causing it to undergo a certain degree of flexure, and to assume the form represented in *Fig.* 140. If two planes, auv, $a'u'v'$, be drawn infinitely near to each other, and normal to the curve Auu'B assumed by the fibres whose lengths remain invariable, such planes will include between them an elementary portion of the solid, and if the system be supposed in equilibrio, the state of equilibrium will not be dis-

turbed by regarding the portion of the solid included between the sections ACD and *uav* as absolutely immoveable, and the portion of the solid included between the sections BEF and *uav* as constituting a distinct system. The conditions of equilibrium in this system will evidently require that the force P, together with the force necessary to retain the part DCA*uav* in its position, shall be just capable of sustaining the efforts arising from the compressions and extensions of the fibres, or, in other words, that all these forces should reduce to two that are equal to each other and directly opposite.

306. If we assume any two rectangular axes Ax and Ay situated in the vertical plane passing through the axis of the solid, we can resolve each of the several forces into two components respectively parallel to these axes; since these forces are all situated in planes parallel to the plane of the axes. Moreover, since the solid has been supposed to be symmetrically divided by the vertical plane passing through the axis, the forces of elasticity arising from the extensions or compressions of the different fibres will be symmetrically disposed with respect to this plane, and the conditions of equilibrium will therefore be the same as though the forces were all situated in this plane. These conditions are, 1°. That the sum of the components parallel to each axis shall be equal to zero; and, 2°. That the sum of the moments of all the forces taken with respect to any line perpendicular to the plane of the forces shall be equal to zero.

307. We shall assume the origin of co-ordinates at the fixed extremity A of the solid, and refer the points in the curve Auu'B to the axes of x and y, which are respectively horizontal and vertical.

308. The normal plane *auv* intersects the cylindrical surface which contains the fibres of an invariable length, and the vertical plane passing through the axis of the solid, in two lines *au* and *uv*, at right angles to each other; and the points in the section *auv* will be referred to two rectangular axes, one of which *au* will be called the axis of u, and the other, parallel to *uv*, and passing through the origin a, will be designated as the axis of v. Thus the two co-ordinates of the point m will be $ao=u$, and $om=v$. The moments of the sev-

eral forces will be referred to the line au, which is frequently called the axis of equilibrium.

309. This being premised, we shall denote by

A and B, the coefficients of elasticity and tenacity (Arts. 298 and 301),

R, the radius of curvature ur, of the curve of flexure, at the point u,

s, the length of the arc Au of the curve of flexure,

x and y, the co-ordinates Ap and pu of the point u referred to the origin A,

x' and y', the co-ordinates of the point B, referred to the same origin,

U and U′, functions of the absciss $ao=u$, expressing the values of the corresponding ordinates ol and ol' of the curve of intersection, reckoned from the axis of equilibrium au, towards the convex and concave sides of the solid,

a, the dimension of the solid estimated along the axis of equilibrium,

V, the greatest value of U or U′, or the distance from the axis of equilibrium to that fibre which is most stretched or compressed at the instant of rupture.

Then, if we consider an elementary portion of the solid, included between the consecutive normal planes, whose base is represented by the element $mm''=du.dv$, of the normal section auv, its original length will be equal to $uu'=ds$; and after the flexure, this length will be increased or diminished, according to its position with reference to the axis of equilibrium, and will be represented by mm' or nn' (*Fig.* 141). But from similarity of the figures rmm', ruu', rnn', we have the proportion

$$ru : rm : rn :: uu' : mm' : nn';$$

or,

$$R : R+v : R-v :: uu' : mm' : nn';$$

and therefore,

$$R : v : v :: uu' : mm'-uu' : uu'-nn' :$$

whence,

$$mm'-uu'=uu'-nn'=\frac{v.uu'}{R}=\frac{v.ds}{R}.$$

This expression will represent the variation in the length of the element whose base is equal to $du.dv$ (Fig. 140) and whose original length was equal to ds. To determine the resistance opposed by this element when thus extended or compressed, we employ the expression (143 *b*) in which we replace na'', the variation in length, by $mm'-uu'$, or $uu'-nn'$; the transverse section m, by $du.dv$; and the length n, by ds: we shall thus obtain an expression for the resistance P'' opposed by the element,

$$P''=\frac{vds}{R}\times\frac{dudvA}{ds}=\frac{Avdvdu}{R}\ldots\ldots(143\,c):$$

and the moment of this resistance taken with reference to the axis of equilibrium au, will be

$$\frac{A}{R}vdvdu\times v=\frac{A}{R}v^2dvdu\ldots\ldots(143\,d).$$

310. The other elementary portions of the solid included between the consecutive normal planes will give similar expressions for the resistances and their moments; and by taking the sums of these expressions, we shall obtain the value of the entire resistance, and that of its moment with reference to the axis au. To determine the value of these sums, we must integrate the expressions (143 *c*) and (143 *d*) throughout the limits of the section auv. This integration is effected, first with reference to one of the variables, v for example; and its value being then substituted in terms of u, we integrate a second time with reference to the other variable. The limits of the first integration will evidently be $v=0$, and $v=U$, for those fibres which suffer extension; and $v=0$, $v=U'$, for those which suffer compression. The limits of the second integration will be $u=0$, and $u=a$.

311. This being premised, the sum of the resistances arising from the extensions of the several fibres will be expressed by

$$\frac{A}{R}\int_0^a du\int_0^U vdv\ldots\ldots(143\,e),^*$$

---

* An expression of the form $\int_0^a du$ is intended to indicate that the integral of du is to be taken between the limits $u=0$, and $u=a$. In like manner, $\int_0^U vdv$ signifies that the integral of vdv should be taken between the limits $v=0$, and $v=U$.

and the sum of the resistances arising from the compressions of the fibres will be

$$\frac{\mathrm{A}}{\mathrm{R}}\int_0^a du\int_0^{\mathrm{U}'} vdv\ \ldots\ldots\ (143\,f).$$

The sum of the moments of these resistances, taken with reference to the axis *au*, will be

$$\frac{\mathrm{A}}{\mathrm{R}}\left(\int_0^a du\int_0^{\mathrm{U}} v^2\,dv+\int_0^a du\int_0^{\mathrm{U}'} v^2\,dv\right)\ \ldots\ldots\ (143\,g).$$

312. For the purpose of resolving the resistances (143 *e*) and (143 *f*) into components parallel to the axes of x and y, we must multiply them respectively by $\frac{dx}{ds}$ and $\frac{dy}{ds}$, the cosines of the angles which their directions form with the axes: but as the curvature assumed by the solid is always found to be exceedingly small even at the instant when the rupture takes place, the expression $\frac{dx}{ds}$ will be very nearly equal to unity, and the components in the direction of the axis of x may therefore be assumed equal to the entire resistances. These being the only forces in the system which have components parallel to the axis of x, the condition of equilibrium which requires that the sum of the components parallel to this axis shall be equal to zero, will be expressed by the equation

$$\int_0^a du\int_0^{\mathrm{U}} vdv-\int_0^a du\int_0^{\mathrm{U}'} vdv=0\ \ldots\ldots\ (143\,h).$$

The negative sign is given to the resistances offered by those fibres which suffer compression, because they are exerted in a direction contrary to the resistances of the extended fibres.

This equation will determine the position of the axis of equilibrium *au* when the figure of the transverse section is known.

313. A similar condition may be obtained for the components parallel to the axis of y; but as it will not be required in the succeeding steps of this investigation, it will be unnecessary to express it analytically.

314. The moment of the force P taken with reference to the axis *au* will be expressed by $\mathrm{P}(x'-x)$, and since this force tends to turn the system about the axis *au*, in a direc-

tion contrary to that in which the resistances of the fibres would cause it to turn, the condition that the algebraic sum of the moments of all the forces taken with reference to the axis of equilibrium shall be equal to zero, will be expressed by the equation

$$\frac{\mathrm{A}}{\mathrm{R}}\left(\int_0^a du\int_0^{\mathrm{U}} v^2 dv+\int_0^a du\int_0^{\mathrm{U}'} v^2 dv\right)-\mathrm{P}(x'-x)=0\,.(143\,i).$$

315. When the radius of curvature becomes equal to unity the expression (143 *g*) becomes

$$\mathrm{A}\left(\int_0^a du\int_0^{\mathrm{U}} v^2 dv+\int_0^a du\int_0^{\mathrm{U}'} v^2 dv\right)\,\ldots\ldots(143\,k).$$

This quantity is called the *moment of elasticity* of the solid, and will depend upon the elasticity of the substance, and the figure of the transverse section. Its value will evidently determine that of the force P, which, acting at the extremity of a given arm of lever, will be necessary to produce a given curvature in the solid; thus, the moment of elasticity becomes a proper measure of the resistance to flexure opposed by the solid.

316. If the flexure of the solid be supposed such that the extreme fibre, or that which undergoes the greatest extension or compression, is about to be ruptured or crushed, the resistance opposed by this fibre will be that due to the tenacity of the substance: hence, if *dudv* denote, as in Art. 309, the base of an elementary portion of the solid included between the consecutive normal sections, and if the distance of this element of the solid from axis of equilibrium *au* be equal to V, that of the fibre which is most extended or compressed, the resistance opposed by such element will be expressed by

$$\mathrm{B}dudv\,;$$

B denoting the coefficient of the tenacity.

This element being at the distance V from the axis of equilibrium, its original length *ds* will undergo a variation represented (Art. 309) by

$$\frac{\mathrm{V}ds}{\mathrm{R}};$$

and the corresponding variation in the length *ds* of the element, whose distance from the same axis is denoted by v, will be

$$\frac{vds}{R}.$$

But the resistances opposed by the two elements being by hypothesis (Art. 293), proportional to their extensions or compressions, we shall have the proportion

$$\frac{Vds}{R} : \frac{vds}{R} :: Bdudv : P'',$$

P″ denoting the resistance opposed by the element at the distance v from the axis of equilibrium. From this proportion we deduce

$$P'' = \frac{v}{V} Bdudv.$$

317. Similar expressions may be obtained for the resistances offered by the other elements; and by taking their moments with reference to the axis of equilibrium, and adding them into one sum, we shall obtain for the moment of the entire resistance, at the instant when a fracture commences,

$$\frac{B}{V}\left(\int_0^a du \int_0^U v^2 dv + \int_0^a du \int_0^{U'} v^2 dv\right) \ldots\ldots (143\ l).$$

This expression is called the *moment of rupture*, and will depend upon the tenacity of the substance, and the figure of the transverse section. This moment must evidently be equal to the moment $P(x'-x)$ of the force P, which is just capable of causing rupture. Thus, we shall have

$$\frac{B}{V}\left(\int_0^a du \int_0^U v^2 dv + \int_0^a du \int_0^{U'} v^2 dv\right) = P(x'-x) \ldots (143m).$$

The value of the moment of rupture will serve to determine that of the force P, which, acting at the extremity of a given arm of lever, will be just capable of producing fracture. Thus, the moment of rupture becomes a proper measure of the resistance to fracture opposed by the solid.

318. By comparing the expression (143 k), for the moment of elasticity, with (143 l), which represents the moment of rupture, we shall perceive that the latter may be deduced from the former by merely substituting $\frac{B}{V}$ for A.

319. When the transverse section can be divided symmetrically by a horizontal line, that line will be the axis of equilibrium, since the equation (143 *h*) will evidently be satisfied by regarding that line as the axis of *u*. The moment of elasticity will then be expressed by

$$2A\int_0^a du\int_0^U v^2 dv;$$

and the moment of rupture by

$$\frac{2B}{V}\int_0^a du\int_0^U v^2 dv,$$

320. In other cases, it will be necessary to determine the position of the axis of equilibrium by the condition (143 *h*), and then to calculate separately the two integrals which enter into the expressions for the moments of elasticity and rupture.

321. To apply these principles, we shall determine the moments of elasticity and rupture for those solids whose transverse sections are such as are more commonly adopted in practice.

322. Let the transverse section be a rectangle (*Fig.* 142), whose breadth and height are denoted respectively by *a* and *b*. The value of the moment of elasticity will then become

$$2A\int_0^a du\int_0^{\frac{1}{2}b} v^2 dv.$$

and by integrating with reference to $v=om$, between the limits $v=0$, and $v=ot=\frac{1}{2}b$, we shall obtain double the sum of the moments of all the elements, whose bases constitute the elementary rectangle *oq*. Performing the integration, we have

$$2A\int_0^a du\times\frac{v^3}{3}=2A\int_0^a du\times\frac{(\frac{1}{2}b)^3}{3}=\frac{1}{12}Ab^3\int_0^a du.$$

Integrating a second time, with reference to *u*, between the limits $u=0$ and $u=a$, we shall obtain for the moment of elasticity ϵ,

$$\epsilon=\frac{1}{12}Aab^3 \;.\;.\;.\;.\;. \;(143\,n).$$

Hence it follows that the resistance to flexure opposed by a solid whose transverse section is rectangular, will be proportional to the breadth and the cube of the depth.

323 If we replace A in this expression by $\frac{B}{V}$, we shall

obtain the moment of rupture β of the rectangle; and since V is in the present case equal to $\frac{1}{2}b$, we shall have

$$\beta=\tfrac{1}{6}Bab^2 \;.\;.\;.\;.\;.\; (143\,o).$$

Thus the resistance to fracture is proportional to the breadth and the square of the depth.

324. If the solid be disposed in such manner that the dimension a shall become vertical, and the dimension b horizontal, the expressions for the moments of elasticity and rupture will become respectively

$$\alpha'=\tfrac{1}{12}Aba^3, \quad \beta'=\tfrac{1}{6}Bba^2;$$

and by comparing these expressions with those obtained, when the dimension a was supposed horizontal, we shall deduce the proportions

$$\alpha : \alpha' :: ab^3 : ba^3 :: b^2 : a^2,$$
$$\beta : \beta' :: ab^2 : ba^2 :: b : a.$$

It thus appears that the resistance to flexure when the broader face b is placed vertically, will be to that exerted when the narrower face a is vertical, as the square of the broader face to the square of the narrower. But that the resistances to fracture in similar cases are proportional simply to the first powers of the same quantities.

325. If in the expressions (143 n) and (143 o), we make $a=b$, we shall obtain for the moments of elasticity and rupture of a prism with a square base,

$$\alpha=\tfrac{1}{12}Aa^4, \quad \beta=\tfrac{1}{6}Ba^3 \;.\;.\;.\;.\;.\; (143\,p).$$

326. Let the transverse section of the solid be a rhombus, (*Fig.* 143), whose diagonals are represented by $2p$ and $2q$, and let the diagonal $2q$ be placed vertically. If we first determine the moment of elasticity of the triangle aBC, that of the rhombus can be immediately deduced by simply multiplying by the number 2. The limits between which the first integration with reference to the variable $v=om$, should be effected, are $v=0$, and $v=ot$. But from the similarity of triangles, we have the proportion

$$ao : ot :: a\text{D} : \text{DC},$$

or,

$$u : ot :: p : q;$$

whence,

$$ot=\frac{uq}{p};$$

and the limits of the first integration will therefore be $v=o$, and $v=\frac{uq}{p}$. Making these substitutions in the general formula for the moment of elasticity, we shall obtain

$$2A\int_0^a du\int_0^{\frac{uq}{p}} v^2dv=2A\int_0^a du\times\tfrac{1}{3}\left(\frac{uq}{p}\right)^3=\tfrac{2}{3}A\frac{q^3}{p^3}\int_0^a u^3du.$$

Integrating a second time, with reference to the variable u, between the limits $u=o$, and $u=p$, the moment of elasticity of the triangle aBC becomes

$$\tfrac{2}{3}A\frac{q^3}{p^3}\int_0^p u^3du=\tfrac{2}{3}A\frac{q^3}{p^3}\times\frac{p^4}{4}=\tfrac{1}{6}Apq^3;$$

and by doubling this expression, we find for the moment of elasticity ϵ of the rhombus,

$$\epsilon=\tfrac{1}{3}Apq^3.$$

327. If in this expression we replace A by $\frac{B}{V}$, we shall obtain the value of the moment of rupture β, which, since $V=q$, will become

$$\beta=\tfrac{1}{3}\frac{B}{q}\times pq^3=\tfrac{1}{3}Bpq^2.$$

328. If we make $p=q$, the rhombus will become a square, and the values of ϵ and β will reduce to

$$\epsilon=\tfrac{1}{3}Ap^4,\quad \beta=\tfrac{1}{3}Bp^3:$$

or if the side of the square be denoted by a, we shall have the relation $a^2=2p^2$, and therefore

$$\epsilon=\tfrac{1}{3}A\times\frac{a^4}{4}=\tfrac{1}{12}Aa^4,\quad \beta=\tfrac{1}{3}B\times\frac{a^3}{2\sqrt{2}}=\frac{1}{6\sqrt{2}}Ba^3;$$

and by comparing these expressions with those obtained (Art. 325) for the moments of elasticity and rupture of a prism with a square base, when the sides of the base are respectively vertical and horizontal, we shall find that the resistance to flexure will be the same whether the diagonal or side of the square be disposed vertically; but that the resistance to fracture when the side is vertical, will be greater than when the diagonal is vertical, in the ratio of $\sqrt{(2)}$ to 1.

329. When the section is a circle whose radius is equal to r, the integration with reference to the variable v must be effected between the limits $v=0$, and $v=\sqrt{(2ru-u^2)}$; and the second integration, with reference to u, between the limits $u=0$, and $u=2r$. Thus, the expression for the moment of elasticity will be

$$\alpha=2\mathrm{A}\int_0^{2r}du\int_0^{\sqrt{(2ru-u^2)}}v^2dv=\tfrac{2}{3}\mathrm{A}\int_0^{2r}(2ru-u^2)^{\frac{3}{2}}du. .(143\,q).$$

For the purpose of effecting the second integration, we make $r-u=z$, which gives

$$du=-dz,\quad 2ru-u^2=r^2-z^2.$$

Substituting these values in the expression for α, and observing that the limits $u=0$, and $u=2r$, correspond to the values $z=+r$, and $z=-r$, we shall obtain

$$\int_0^{2r}(2ru-u^3)^{\frac{3}{2}}du=-\int_{-r}^{r}(r^2-z^2)^{\frac{3}{2}}dz=$$

$$-\int_{-r}^{r}(r^2-z^2)(r^2-z^2)^{\frac{1}{2}}dz;$$

or,

$$\int_0^{2r}(2ru-u^2)^{\frac{3}{2}}du=\int_{-r}^{r}z^2(r^2-z^2)^{\frac{1}{2}}dz$$

$$-\int_{-r}^{r}r^2(r^2-z^2)^{\frac{1}{2}}dz\,\ldots\ldots(143\,r).$$

The first term of the second member, being integrated by parts, gives

$$\int_{-r}^{r}z^2(r^2-z^2)^{\frac{1}{2}}dz=\int_{-r}^{r}\frac{z}{2}(r^2-z^2)^{\frac{1}{2}}2zdz=$$

$$-(r^2-z^2)^{\frac{3}{2}}\cdot\frac{z}{3}+\tfrac{1}{3}\int_{-r}^{r}(r^2-z^2)^{\frac{3}{2}}dz\,\ldots\ldots(143\,s).$$

The quantity $(r^2-z^2)^{\frac{3}{2}}\frac{z}{3}$ will reduce to zero, when $z=+r$, or $z=-r$, this term will therefore disappear; and the last term being resolved into factors will reduce equation (143 s) to

$$\int_{-r}^{r}z^2(r^2-z^2)^{\frac{1}{2}}dz=\tfrac{1}{3}\int_{-r}^{r}(r^2-z^2)^{\frac{1}{2}}r^2-\tfrac{1}{3}\int_{-r}^{r}(r^2-z^2)^{\frac{1}{2}}z^2dz;$$

whence, by transposition and reduction, we obtain

$$\int_{-r}^{r} z^2(r^2-z^2)^{\frac{1}{2}}dz=\tfrac{1}{4}\int_{-r}^{r}(r^2-z^2)^{\frac{1}{2}}r^2dz.$$

This value being substituted in (143 r) gives

$$\int_{0}^{2r}(2ru-u^2)^{\frac{3}{2}}du=-\tfrac{3}{4}r^2\int_{-r}^{r}(r^2-z^2)^{\frac{1}{2}}dz.$$

But the integral $\int_{-r}^{r}(r^2-z^2)^{\frac{1}{2}}dz$ represents the area of a semicircle whose radius is equal to r. This area being expressed by $\frac{1}{2}\pi r^2$, we shall have

$$\int_{0}^{2r}(2ru-u^2)^{\frac{3}{2}}du=-\tfrac{3}{8}\pi r^4;$$

and by substituting this value in the expression (143 q) for the moment of elasticity α, it will become

$$\alpha=\tfrac{1}{4}A\pi r^4.$$

330. To determine the moment of rupture β, we replace A by $\frac{B}{V}$ or $\frac{B}{r}$, and thus obtain

$$\beta=\tfrac{1}{4}B\pi r^3.$$

331. By comparing these values with the expressions (143 p), we shall find that the moments of elasticity and rupture of a square are to those of the inscribed circle as 1 to $\frac{3\pi}{16}$.

332. The moment of elasticity of a tube or hollow cylinder whose exterior and interior diameters are represented by r' and r'', will be determined by taking the difference of the moments of the exterior and interior sections. Thus we shall have

$$\alpha=\tfrac{1}{4}A\pi(r'^4-r''^4),$$

and the moment of rupture β will be found by replacing A by $\frac{B}{V}$ or $\frac{B}{r'}$; hence,

$$\beta=\tfrac{1}{4}B\pi\frac{r'^4-r''^4}{r'}.$$

333. If the section of the hollow cylinder be supposed equal to that of a solid cylinder, the radius of the latter being denoted by r, we shall have the relation

$$r^2=r'^2-r''^2;$$

and the resistances to fracture opposed by the two will be to each other as

$$\frac{r'^4-r''^4}{r'}:(r'^2-r''^2)^{\frac{3}{2}}, \text{or as } \frac{r'^2+r''^2}{r'}:(r'^2-r''^2)^{\frac{1}{2}};$$

replacing $r'^2-r''^2$ by its value r^2, this ratio will be reduced to

$$r'+\frac{r''^2}{r'}:r.$$

The first term of this ratio must always exceed the second: thus the resistance to fracture opposed by the hollow cylinder will always be greater than that offered by the solid cylinder; and since the value of the first term may be increased indefinitely without affecting that of the second, it follows that the resistance of the hollow cylinder may likewise be increased indefinitely without changing the area of its section.

334. Let a and b represent the breadth and height of a rectangle inscribed in a circle whose diameter is denoted by D: we shall have the relation $a^2+b^2=\mathrm{D}^2$; and therefore,

$$ab^2=a(\mathrm{D}^2-a^2).$$

But the moment of rupture of a rectangle being proportional to the breadth and the square of the depth (Art. 323), if we wish the resistance to fracture to be a maximum, we must differentiate the preceding expression with reference to a, and place the first differential coefficient equal to zero: we shall thus obtain

$$\frac{d(ab^2)}{da}=\mathrm{D}^2-3a^2=0;$$

and therefore,

$$a^2=\tfrac{1}{3}\mathrm{D}^2, \quad b^2=\mathrm{D}^2-a^2=\tfrac{2}{3}\mathrm{D}^2.$$

Hence, the strongest rectangular solid which can be cut from a given cylinder will be that in which the diameter of the cylinder, the depth of the rectangular section, and its breadth, shall be to each other as the square roots of the numbers 3, 2, and 1.

Of the Figure of the Solid after Flexure.

335. We will now consider the form of the curve $Auu'B$ (*Fig.* 140) assumed by the fibres whose lengths remain invariable. For this purpose, let AM (*Fig.* 144) represent the solid which is firmly fixed at its extremity A, and subjected to the action of the weight P, applied at the other extremity, in a direction perpendicular to the original direction of the axis of the solid. Then denoting by ε the moment of elasticity, the equation (143 *i*), which expresses a condition of equilibrium, when the solid merely undergoes flexure, without being ruptured, will become

$$\frac{\varepsilon}{R}=P(x'-x);$$

or by substituting for the radius of curvature R its general value $\dfrac{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}$, this equation will reduce to

$$\varepsilon\frac{\dfrac{d^2y}{dx^2}}{\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}}}=P(x'-x)\ .\ .\ .\ .\ .\ (143\,t).$$

336. In like manner, when the solid is about to be ruptured, if we substitute β for the moment of rupture, in equation (143 *m*), we shall obtain

$$\beta=P(x'-x)\ .\ .\ .\ .\ .\ (143\,u).$$

337. Let c denote the horizontal distance AB between the extremities of the solid,

f, the ordinate BM,

s, the length of the arc AmM,

ω, the angle included between the tangent to the curve at the point M and the horizontal line.

Then, since the curvature is supposed to be extremely small, even at the instant when fracture takes place, the expression $\dfrac{dy}{dx}$, which represents the tangent of the angle formed by the element of the curve with the axis of x, will also be extremely small, and its square may therefore be neglected in comparison with unity. Thus the equation (143 t) will be reduced to

$$\varepsilon\frac{d^2y}{dx^2}=\mathrm{P}(c-x).$$

Multiplying by dx, we obtain

$$\varepsilon\frac{d^2y}{dx^2}dx=\mathrm{P}(e-x)dx\,;$$

and by integration, we have

$$\varepsilon\frac{dy}{dx}=\mathrm{P}\left(cx-\frac{x^2}{2}\right)\ \ldots\ldots\ (143\ v).$$

The arbitrary constant introduced by integration is equal to zero; since, when $x=0$, $\frac{dy}{dx}$, which represents the tangent of the angle included between the element of the curve and the axis of abscisses, is likewise equal to zero.

Multiplying again by dx we have

$$\varepsilon\frac{dy}{dx}dx=\mathrm{P}\left(cx-\frac{x^2}{2}\right)dx\,;$$

and performing a second integration, there results

$$\varepsilon y=\mathrm{P}\left(\frac{cx^2}{2}-\frac{x^3}{6}\right):$$

the constant will be equal to zero, since $x=0$ gives $y=0$.

338. If in this expression we make $x=c$, the ordinate y will become equal to f; hence we shall have

$$f=\frac{\mathrm{P}}{\varepsilon}\left(\frac{c^3}{2}-\frac{c^3}{6}\right)=\frac{\mathrm{P}}{\varepsilon}\times\frac{c^3}{3}\ \ldots\ldots\ (143\ w).$$

In like manner, by making $x=c$ in equation (143 v), we shall have $\frac{dy}{dx}=$ tang α, and therefore

$$\text{tang}\ \alpha=\frac{\mathrm{P}}{\varepsilon}\left(e^2-\frac{c^2}{2}\right)=\frac{\mathrm{P}}{\varepsilon}\times\frac{c^2}{2},$$

or, replacing $\frac{\mathrm{P}}{\varepsilon}$ by its value $\frac{3f}{c^3}$ deduced from the preceding equation, we have

$$\text{tang}\ \alpha=\frac{3f}{2c}\ \ldots\ldots\ (143\ x).$$

339. To determine the length s of the arc AmM, we take the general expression for the element ds of this arc,

$$ds=\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}dx,$$

which, being developed, rejecting all but the two first terms as inconsiderable, gives

$$ds=dx\left(1+\frac{dy^2}{2dx^2}\right);$$

and by replacing $\frac{dy}{dx}$ by its value (143 v), this equation becomes

$$ds=dx+\tfrac{1}{2}dx(c^2x^2-cx^3+\tfrac{1}{4}x^4)\frac{\mathrm{P}^2}{\varepsilon^2}.$$

Integrating, we obtain

$$s=x+\frac{\mathrm{P}^2}{\varepsilon^2}\left(\frac{c^2x^3}{6}-\frac{cx^4}{8}+\frac{x^5}{40}\right);$$

and by making $x=c$, the value of the entire arc AmM becomes

$$s=c+\frac{\mathrm{P}^2}{\varepsilon^2}\left(\frac{c^5}{6}-\frac{c^5}{8}+\frac{c^5}{40}\right)=c+\frac{\mathrm{P}^2}{\varepsilon^2}\times\frac{c^5}{15},$$

or, replacing $\frac{\mathrm{P}^2}{\varepsilon^2}$ by its value deduced from equation (143 w), this expression reduces to

$$s=c+\frac{3f^2}{5c}\;\ldots\ldots\;(143\;y).$$

340. When the weight P is just sufficient to fracture the solid, the rupture will take place at the supported end; since the moment P$(c-x)$ of the force P will be the greatest when $x=0$: the equation (143 u) will then become

$$\beta=\mathrm{P}c\;\ldots\ldots\;(143\;z);$$

or, if the curvature be still supposed so small that $\frac{dy^2}{dx^2}$ may be neglected in comparison with unity, the equation of the curve will be the same as when the flexure was extremely slight, and we shall therefore have

$$\mathrm{P}=\frac{\beta}{s-\frac{3f^2}{5c}}.$$

341. Let it now be supposed that the solid is loaded with

weights distributed uniformly throughout its length. Denote by z the absciss of any point between M and m, and by p the weight supported by a portion of the solid which corresponds to a unit of length of the absciss: then since the distribution of the weights is supposed uniform, we shall have the proportion

$$1 : p :: dz : pdz,$$

the weight supported by the element of the solid whose projection on the axis of x is represented by dz. The moment of this weight, with reference to the point m, will be $pdz(z-x)$, and the sum of the moments of all the weights supported between M and m, taken with reference to the same point m, will be

$$\int p(z-x)dz.$$

This integral should be taken between the limits $z=c$ and $z=x$, the quantity x being regarded as invariable: thus we shall have

$$\int_x^c p(z-x)dz = p\frac{c^2-x^2}{2} - px(c-x) \quad \ldots\ldots (143\ a').$$

But the condition of equilibrium requires that the sum of these moments shall be equal to $\varepsilon\frac{d^2y}{dx^2}$, the sum of the moments of the resistances offered by the several fibres. Hence, we obtain

$$\varepsilon\frac{d^2y}{dx^2} = p\left(\frac{c^2-x^2}{2}\right) - px(c-x) = \tfrac{1}{2}pc^2 - pcx + \tfrac{1}{2}px^2.$$

Multiplying by dx, and integrating, we obtain

$$\varepsilon\frac{dy}{dx} = p(\tfrac{1}{2}c^2x - \tfrac{1}{2}cx^2 + \tfrac{1}{6}x^3);$$

and multiplying a second time by dx, and integrating, there results

$$\varepsilon y = p(\tfrac{1}{4}c^2x^2 - \tfrac{1}{6}cx^3 + \tfrac{1}{24}x^4).$$

Making $x=c$, $y=f$, and $\frac{dy}{dx}=\text{tang}\ \alpha$, we find

$$f = \frac{p}{\varepsilon}(\tfrac{1}{4}c^4 - \tfrac{1}{6}c^4 + \tfrac{1}{24}c^4) = \frac{p}{\varepsilon}\cdot\frac{c^4}{8} \quad \ldots\ldots (143\ b'),$$

$$\text{tang}\ \alpha = \frac{p}{\varepsilon}(\tfrac{1}{2}c^3 - \tfrac{1}{2}c^3 + \tfrac{1}{6}c^3) = \frac{4f}{3c}.$$

342. When the weights distributed along the solid are just capable of producing rupture, the fracture will take place at the supported end, since the expression (143 a') which represents the sum of the moments of these weights will evidently be the greatest when $x=0$. This sum being then equal to the moment of rupture β, we shall have

$$\beta=\tfrac{1}{2}pc^2, \quad cp=\frac{2\beta}{c} \ldots\ldots (143\ c');$$

the expression pc represents the entire weight distributed along the solid.

343. If we make $cp=\mathrm{P}$, and compare the values (143 b') and (143 w) of the ordinate f, it will appear that the depression of the point M below the horizontal line Ax, produced by the action of the weight P applied at the point M, will be greater than the depression produced by an equal weight distributed uniformly along the solid, in the ratio of 8 to 3. And by comparing the values of $\frac{\beta}{c}$ in equations (143 c') and (143 z) we shall perceive that the weight necessary to fracture the solid, when distributed uniformly, will be double that required when it is applied at the extremity M.

344. It frequently occurs that the weight of the solid forms an important part of the load which it is required to sustain. The effect produced by this weight is readily calculated by regarding it as uniformly distributed throughout the solid. Thus, if the solid be loaded with its own weight $\mathrm{P}'=pc$, and a weight P applied at its extremity M, the sum of the moments of the weight P, and the weight of that portion of the solid which lies to the right of the point m, taken with reference to that point, will, by Arts. 337 and 341, be

$$\mathrm{P}(c-x)+p(\tfrac{1}{2}c^2-cx+\tfrac{1}{2}x^2);$$

and in case of equilibrium, we shall have

$$\epsilon\frac{d^2y}{dx^2}=\mathrm{P}(c-x)+p(\tfrac{1}{2}c^2-cx+\tfrac{1}{2}x^2) \ldots\ldots (143\ d');$$

or, if the solid be supposed on the point of being ruptured, the fracture taking place at the point A, for which $x=0$, the condition of equilibrium will be

$$\beta=\mathrm{P}c+\tfrac{1}{2}pc^2.$$

345. The expression (143 d') gives, by two successive integrations,

$$\varepsilon\frac{dy}{dx}=\mathrm{P}\left(cx-\frac{x^2}{2}\right)+p(\tfrac{1}{2}c^2x-\tfrac{1}{2}cx^2+\tfrac{1}{6}x^3),$$

$$\varepsilon y=\mathrm{P}(\tfrac{1}{2}cx^2-\tfrac{1}{6}x^3)+p(\tfrac{1}{4}c^2x^2-\tfrac{1}{6}cx^3+\tfrac{1}{24}x^4);$$

and by making $x=c$, $y=f$, and $\frac{dy}{dx}=$ tang α, we obtain

$$f=\frac{c^3}{\varepsilon}(\tfrac{1}{3}\mathrm{P}+\tfrac{1}{8}pc)=\frac{c^3}{\varepsilon}(\tfrac{1}{3}\mathrm{P}+\tfrac{1}{8}\mathrm{P}'),$$

$$\text{tang }\alpha=\frac{c^2}{\varepsilon}(\tfrac{1}{2}\mathrm{P}+\tfrac{1}{6}pc)=\frac{3\mathrm{P}+\mathrm{P}'}{8\mathrm{P}+3\mathrm{P}'}\cdot\frac{4f}{c},$$

$$\beta=c(\mathrm{P}+\tfrac{1}{2}pc)=c(\mathrm{P}+\tfrac{1}{2}\mathrm{P}')$$

346. When the solid is supported in a horizontal position at its two extremities M and M′ (*Fig*. 145), and loaded with weights at its middle point A, the results obtained Arts. 337–340 will apply to each half of the curve assumed by the solid; for we may regard either half as perfectly immoveable, and suppose the other portion to be solicited by a force acting at its extremity and equal to the resistance offered by one of the points of support. Hence, if we denote by

2P, the weight suspended at the middle point,
2c, the distance between the points of support,
2s, the length of the curve,
f, the sagitta CA,
α, the angle included between the line MM′ and the tangent to the curve at M or M′;

the resistance exerted by each fixed point in the vertical direction will be equal to P, one-half the weight applied at A, and the formulas (143 w), (143 x), (143 y), and (143 z) will become immediately applicable to the present case. Hence,

$$f=\frac{\mathrm{P}}{\varepsilon}\cdot\frac{c^3}{3}=\frac{(2c)^3}{\varepsilon}\cdot\frac{2\mathrm{P}}{48}\;\ldots\ldots\;(143\;e'),$$

$$\text{tang }\alpha=\frac{3f}{2c}$$

$$2s=2c+\frac{6f^2}{5c}$$

$$\beta=c\mathrm{P}\;\ldots\ldots\;(143\;f').$$

The value of f indicates that the depression of the solid at the middle point, or the sagitta AC, will be proportional to the weight 2P, and the cube of the distance between the points of support.

347. The expressions deduced in the preceding article have been obtained upon the supposition that the resistances opposed by the fixed points were exerted in a vertical direction; whereas, the resistance is actually exerted in the direction of the normal to the curve at the point M or M′; and in some instances the inclination of this normal to the vertical line is too great to be neglected. This circumstance will seldom occur except in the case of fracture, the curvature of the solid being then greater than in the case of a mere flexure. If we represent the resistance exerted at M′ by the line M′F, and resolve this force into two components which shall be respectively vertical and horizontal, the latter component ME will be equal and opposite to the similar component of the resistance at the point M, and the vertical component M′D will be equal to P, or to one-half the weight supported at the middle point of the solid. The value of the horizontal component M′E may be readily found; for we have

$$\mathrm{M'E}=\mathrm{DF}=\mathrm{M'D}\times\mathrm{tang}\ \mathrm{DM'F}=\mathrm{P}\,.\,\mathrm{tang}\ \omega.$$

When the equilibrium subsists, and the solid is on the point of being ruptured, the moment of rupture must be equal to the sum of the moments of the vertical and horizontal components. The moment of the former, with reference to the point A, has been found equal to $c\mathrm{P}$; that of the latter will obviously be $\mathrm{P}\ \mathrm{tang}\ \omega\times\mathrm{AC}=\mathrm{P}\ \mathrm{tang}\ \omega\,.\,f$; thus, the conditions of equilibrium will become

$$\rho=\mathrm{P}c+\mathrm{P}\ \mathrm{tang}\ \omega\,.\,f;$$

or, if we suppose the curve to be represented by the same equation as in Art. 337, in which case $\mathrm{tang}\ \omega=\frac{3f}{2c}$, this relation may be written

$$\rho=c\mathrm{P}\left(1+\frac{3f^2}{2c^2}\right).$$

348. If the weight be uniformly distributed throughout the

length of the solid, we may regard each half as firmly fixed at the point A, and solicited at the same time by a system of parallel forces applied at every point of the solid, and acting downwards; and by a single force equal to their sum, or to the resistance offered by the point of support, applied at the extremity of the solid, and acting upwards. Thus, the case will be the same as that considered in Art. 344, with the exception that the forces arising from the weights uniformly distributed along the solid are exerted in contrary directions. The equations obtained in that case will therefore become applicable to the present one by simply changing the signs of the moments of these forces, and replacing P by pc; we shall thus obtain

$$\varepsilon\frac{dy}{dx}=cp(cx-\tfrac{1}{2}x^2)-p(\tfrac{1}{2}c^2x-\tfrac{1}{2}cx^2+\tfrac{1}{6}x^3)\ldots\ldots(143\,g');$$

$$\varepsilon y=cp(\tfrac{1}{2}cx^2-\tfrac{1}{6}x^3)-p(\tfrac{1}{4}c^2x^2-\tfrac{1}{6}cx^3+\tfrac{1}{24}x^4)\ldots\ldots(143\,h');$$

$$\beta=cp\,.\,c-cp\,.\,\tfrac{1}{2}c\ldots\ldots(143\,i');$$

making $x=c$, $y=f$, $\frac{dy}{dx}=$ tang ε, we obtain

$$f=\frac{p}{\varepsilon}(\tfrac{1}{2}-\tfrac{1}{6}-\tfrac{1}{4}+\tfrac{1}{6}-\tfrac{1}{24})c^4=\frac{c^3}{\varepsilon}\cdot\frac{5cp}{24},$$

$$\text{tang}\,\varepsilon=\frac{p}{\varepsilon}(1-\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{6})c^3=\tfrac{1}{3}\frac{pc^3}{\varepsilon}=\frac{8f}{5c},$$

$$\beta=cp\,.\,\tfrac{1}{2}c\ldots\ldots(143\,k').$$

By comparing this value of f with that obtained in equation (143 e'), it will appear that the depression of the solid at its middle point produced by a weight $2pc$ uniformly distributed throughout the solid, will be less than that produced by the same weight suspended at the middle point, in the ratio of 5 to 8. And by comparing the values of $\frac{2\beta}{c}$ given by equations (143 k') and (143 f') we shall perceive that the solid will be equally liable to fracture by the action of the weight $2pc$ distributed uniformly, or by half that weight applied at its middle point.

349. The preceding expressions, like those in Art. 346, have been obtained upon the supposition that the resistances offered by the fixed points are exerted in vertical directions.

In the case of rupture, the line of direction of the resistance may deviate so far from the vertical as to render the above supposition inadmissible. We then resolve this resistance, as in Art. 347, into two components respectively vertical and horizontal; the former will be represented by pc, and the latter by $pc \,.\, \text{tang}\,\alpha$. In case of equilibrium, it will simply be necessary to add to the second member of equation (143 i') the moment $pc \,.\, \text{tang}\,\alpha \times f$, of the horizontal component; thus, we shall have

$$\beta = cp \,.\, c - cp \,.\, \tfrac{1}{2}c + cp \,.\, \text{tang}\,\alpha \,.\, f = cp(\tfrac{1}{2}c + f\,\text{tang}\,\alpha),$$

or, by replacing tang α by its value $\frac{8f}{5c}$, we have

$$\beta = cp \,.\, \tfrac{1}{2}c\left(1 + \frac{16f^2}{5c^2}\right),$$

and therefore,

$$2cp = \frac{4\beta}{c\left(1 + \frac{16f^2}{5c^2}\right)}:$$

we here suppose that the equation of the curve remains the same as in Art. 337.

350. If the solid be loaded at the same time with a weight 2P at its middle point, and its own weight $2pc = 2\text{P}'$ uniformly distributed, the case will be similar to that considered in the two preceding articles, with the exception that the force applied at the extremity of the solid will now be represented by $\text{P} + pc = \text{P} + \text{P}'$: thus, when we suppose the resistances exerted by the fixed points to act vertically, we shall obtain, by substituting $\text{P} + pc$ for pc in the first terms of the second members of equations (143 h') and (143 i'),

$$\epsilon y = (\text{P} + pc)(\tfrac{1}{2}cx^2 - \tfrac{1}{6}x^3) - p(\tfrac{1}{4}c^2x^2 - \tfrac{1}{6}cx^3 + \tfrac{1}{24}x^4),$$

$$\beta = (\text{P} + pc)c - cp \,.\, \tfrac{1}{2}c \;.\;.\;.\;.\;. \;(143\; l');$$

which give, by making $x = c$, $y = f$, and $pc = \text{P}'$,

$$f = \frac{c^3}{3\epsilon}\left(\text{P} + \frac{5\text{P}'}{8}\right), \quad \beta = c(\text{P} + \tfrac{1}{2}\text{P}').$$

351. But, if regard be had to the oblique direction of the resistance, as may be necessary in the case of rupture, we must add the moment of the horizontal component

to the second member of equation (143 l'), which thus becomes

$$\beta=(\mathrm{P}+pc)c-cp\,.\,\tfrac{1}{2}c+(\mathrm{P}+pc)\operatorname{tang}\alpha\,.\,f;$$

and therefore,

$$2\mathrm{P}=\frac{2\beta-\mathrm{P}'(c+2f\operatorname{tang}\alpha)}{c+f\operatorname{tang}\alpha}.$$

The equation (143 g') likewise gives, by replacing pc in the first term of the second member by $\mathrm{P}+pc$, and making $\frac{dy}{dx}=\operatorname{tang}\alpha$,

$$\operatorname{tang}\alpha=\frac{3\mathrm{P}+2\mathrm{P}'}{8\mathrm{P}+5\mathrm{P}'}\cdot\frac{4f}{c};$$

and this value of *tang* α may be regarded as sensibly equal to that employed in the preceding expression for the value of 2P.

352. To apply the several results which have been obtained to particular cases, it will be necessary to substitute the values of the moments of rupture and elasticity appertaining to the figure of the transverse section. We must likewise assign to A and B the coefficients of elasticity and tenacity, their particular values which depend upon the nature of the substance, and which are supposed to have been previously determined by experiment.

353. The best method of determining the values of A and B consists in supporting a prismatic solid at its two extremities in a horizontal position, loading it with weights at its middle point, and observing the sagittæ which correspond to different weights; or simply, the weight and sagitta at the instant when the fracture is about to take place.

If the transverse section of the solid be a rectangle, whose breadth and height are denoted respectively by a and b, we shall have (Arts. 322 and 323),

$$\varepsilon=\tfrac{1}{12}\mathrm{A}ab^3,\quad \beta=\tfrac{1}{6}\mathrm{B}ab^2;$$

and if we neglect the weight of the solid (Arts. 346 and 347),

$$f=2\mathrm{P}\frac{(2c)^3}{48\varepsilon},\quad \beta=c\mathrm{P}\left(1+\frac{3f^2}{2c^2}\right);$$

and by eliminating ε and β, we obtain, for the case of simple flexure,

$$f=2P\frac{(2c)^3}{4Aab^3},\quad \text{or } A=2P\frac{(2c)^3}{4ab^3f}\ldots\ldots(143\ m');$$

and for that of fracture

$$B=2P\frac{3c}{ab^2}\left(1+\frac{3f^2}{2c^2}\right)\ldots\ldots(143\ n');$$

$2c$ being the interval between the supports, and $2P$ the weight with which the solid is loaded.

The values of A and B are thus expressed in functions of quantities which are readily determined by observation.

354. If the weight of the solid $2P'$ be likewise taken into consideration, it will simply be necessary (Art. 350) to add $\frac{5}{8}.2P'$ to $2P$ in equation (143 m'), and to replace equation (143 n') by the formulas of Art. 351: we shall thus have, in the case of flexure,

$$f=(2P+\tfrac{5}{8}.2P')\frac{(2c)^3}{4Aab^3},\quad A=(2P+\tfrac{5}{8}.2P')\frac{(2c)^3}{4ab^3f},$$

and for that of fracture,

$$B=\frac{6s}{ab^2}=\frac{(2P+2P')(c+f.\text{tang}\,\omega)-P'c}{\frac{1}{6}ab^2},$$

$$\text{tang}\,\omega=\frac{3P+2P'}{8P+5P'}\cdot\frac{4f}{c}.$$

355. If the solid be loaded with a weight $2Q$, and if the corresponding sagitta be denoted by f', we shall obtain a value for f' similar to that of f in the preceding article: thus we shall have,

$$f'=(2Q+\tfrac{5}{8}.2P')\frac{(2c)^3}{4Aab^3};$$

and by taking the difference between f and f', the weight of the solid $2P'$ will disappear, and we shall obtain

$$f'-f=(2Q-2P)\frac{(2c)^3}{4Aab^3},\quad A=(2Q-2P)\frac{(2c)^3}{4ab^3(f'-f)}.$$

Thus, it will only be necessary to observe the increase $f'-f$ in the sagitta, which corresponds to a given increase $2Q-2P$ in the weights suspended at the middle point.

Of Solids of equal Resistance.

356. When a solid having the prismatic form is subjected to an effort which tends to break it, there will always be a particular point at which the fracture will be most likely to take place. For, the moment of rupture will be the same at every point, whilst the moment of the force applied will depend upon its distance from the point with reference to which the moments are taken. Hence, if the strength of the solid be sufficient at that point where a rupture is most likely to occur, it will be unnecessarily great at other points.

357. It becomes an object, therefore, to determine the figure of the solid which shall be uniformly strong throughout, since the adoption of such a figure may frequently effect a material reduction in the quantity of materials employed. Solids having such figures are called solids of equal resistance.

358. As an example, let a body ABM (*Fig.* 146), whose upper surface AB is horizontal, and whose two lateral faces are vertical, be firmly fixed at its extremity A, and subjected to the action of a weight P suspended from its other extremity. It is required to determine the form of the under surface BmM such that the solid may be equally strong throughout, or that the moment of the weight P taken with reference to any point in the length of the solid, shall be equal to the moment of rupture of the transverse section at the same point.

Denote by a the breadth of the solid, b the height AM, c the length AB, x the variable absciss Bp, and v the corresponding ordinate pm: the moment of rupture of the section AM will be (Art. 323) $B\frac{ab^2}{6}$; and since this must be equal to the moment of the force P, we shall have

$$Pc = B\frac{ab^2}{6}, \quad b = \sqrt{\left(\frac{6Pc}{Ba}\right)}.$$

In like manner, the moment of rupture of the section pm will be $B\frac{av^2}{6}$, and the moment of the force P with reference to a point in this section will be Px. These moments being equal by the conditions of the problem, the general relation between the quantities v and x will become

$$Px = B\frac{av^2}{6}, \quad v^2 = \frac{b^2x}{c}.$$

This equation evidently appertains to a parabola, the axis of which will be the line AB.

359. To determine the figure of the curve assumed by the solid when bent, we observe that the moment of elasticity of the section pm will be (Art. 322) $A\frac{av^3}{12} = A\frac{ab^3x^{\frac{3}{2}}}{12c^{\frac{3}{2}}}$. Hence, if y denote the ordinate of the curve of flexure corresponding to the absciss $Ap = c - x$, the conditions of equilibrium in case of flexure will be (Art. 337)

$$A\frac{ab^3x^{\frac{3}{2}}}{12c^{\frac{3}{2}}} \times \frac{d^2y}{dx^2} = Px.$$

Performing two successive integrations, and remarking that when $x = c$, $\frac{dy}{dx} = 0$, and $y = 0$, we obtain

$$\frac{dy}{dx} = \frac{P}{A} \cdot \frac{24c^{\frac{3}{2}}}{ab^3}\left(x^{\frac{1}{2}} - c^{\frac{1}{2}}\right), \quad y = \frac{P}{A} \cdot \frac{24c^{\frac{3}{2}}}{ab^3}\left(\tfrac{2}{3}x^{\frac{3}{2}} - c^{\frac{1}{2}}x + \tfrac{1}{3}c^{\frac{3}{2}}\right);$$

and by making $x = 0$, and $y = f$, we find, for the depression of the extreme point B,

$$f = \frac{P}{A} \cdot \frac{8c^3}{ab^3}.$$

By comparing this expression with that obtained in equation (143 w), it will appear that the depression f is twice as great in the present instance as when the solid had the prismatic form.

360. If the weight supported by the solid be distributed uniformly along its length, each unit of length being supposed to support a weight p, the sum of the moments of these weights, taken with reference to the point A, will be (Art. 342) $pc \cdot \frac{1}{2}c$; and the condition of equilibrium will therefore be

$$B\frac{ab^2}{6} = pc \cdot \tfrac{1}{2}c.$$

In like manner, the sum of the moments of the weights supported between the points p and B, taken with reference to the point p, will be $px \cdot \frac{1}{2}x$. Hence, we shall have

$$B\frac{av^2}{6} = px \cdot \frac{1}{2}x, \quad \text{or } v = \frac{bx}{c},$$

the equation of a right line.

361. The preceding examples will be sufficient to illustrate the manner in which the form of the solid of equal resistance may be determined when the distribution of the load is previously known.

Of the Principle of Virtual Velocities.

362. The principle of virtual velocities, which was discovered by Galileo, and very fully developed by John Bernouilli and Lagrange, may frequently prove of great utility in stating the analytical conditions of statical problems. Indeed, it is regarded by Lagrange, who has adopted it as the basis of his "Mecanique Analytique," as so essential, that he considers all the general methods which can be employed in the solution of questions relating to equilibrium, as being nothing more than applications more or less direct of this general principle.

363. *A virtual velocity* is the path described by the point of application of a force, when the equilibrium is disturbed in an infinitely small degree. Thus, by supposing that the point of application m of a force P (*Fig.* 147) is, by an instantaneous derangement of the system, transferred to n, the small line mn which it describes is called the virtual velocity of the point m.

364. If this virtual velocity be projected upon the direction of the force, it will occupy thereon the small space ma, and the product of the force P by this projection ma is called the moment of this virtual velocity, or, sometimes, the moment of the force; it should however be observed, that the term moment is here employed with a very different signification from that usually implied.

The principle of virtual velocities, as will be demonstrated, consists in this, that when the system is in equilibrio, the sum of these moments is equal to zero; thus, if P, P′, P″, &c., represent different forces applied to a system, and p, p', p'', &c., the projections of the virtual velocities on the directions of these forces, we must have in case of equilibrium,

$$Pp+P'p'+P''p''+\&c.=0\ .\ .\ .\ .\ .\ (144).$$

It is necessary to remark that when any one of these projections p, p', p'', &c., falls upon the prolongation mb (*Fig.* 148) of the force P, applied at m, this projection must be regarded as negative; and since the forces P, P′, P″, &c., are all considered as having the positive sign, the moment corresponding to this negative projection, must likewise be affected with the negative sign; thus, the equation (144) will express that the algebraic sum of the moments is equal to zero.

365. This principle will first be demonstrated for that case in which the forces are applied to a single point. Let P, P′, P″, &c., represent any number of forces applied to the point m (*Fig.* 149), and sustaining it in equilibrio; if, by the effect of an infinitely small derangement, the point m be transported to n, the line mn being infinitely small, may be regarded as a right line. Let the axis of x be supposed to coincide in direction with the line mn, and denote by α, α', α'', &c., the angles formed by the several forces with this axis; we shall have, since an equilibrium subsists in the system,

$$P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''+\&c.=0:$$

multiplying the several terms of this equation by the line mn, which will be denoted by z, we shall obtain

$$Pz\cos\alpha+P'z'\cos\alpha'+P''z''\cos\alpha''+\&c.=0\ .\ .\ .\ .\ .\ (145).$$

But it is evident that $z\cos\alpha$, or $mn\,.\cos nml$, is equal to the small line ml, the projection of mn on the direction of the force P. Thus $z\cos\alpha$ represents the same quantity as the letter p in equation (144). The same remarks being applicable to the other forces, the several products $z\,.\cos\alpha'$, $z\cos\alpha''$, &c., may be replaced by p', p'', &c., the projections of the virtual velocity of the point m upon the directions of these forces, and the equation (145) will then become

$$Pp+P'p'+P''p''+\&c.=0\,;$$

from which we conclude that the principle of virtual velocities is true when the forces are applied to a single point.

366. The most general case of this principle which usually presents itself, is that in which the several forces P, P′, P″, &c., are applied to different points of a body or system of bodies: these points preserving their distances invariable; may be regarded as connected with each other by inflexible right lines. Before examining the general state of the system when the equilibrium has been slightly disturbed, we will consider singly one of these inflexible right lines *mm′*, at the instant when the point *m* has been brought into the position denoted by *n*. The other extremity *m′* of this right line will at the same time change its position, and may be situated either above *mm′* (*Fig.* 150), or beneath it (*Fig.* 151): let it be first supposed above *mm′*, and the line *mm′* will then assume the position *nn′* (*Fig.* 152): the lines *mn* and *m′n′* may be regarded as infinitely small when compared with the lines *mm′* and *nn′*, since the derangement of the system is supposed infinitely small. If the points *m* and *n′* be connected by a right line we shall form a triangle *mm′n′*, in which the side *m′n′* being infinitely small, the angle *n′mm′* will likewise be infinitely small, and the arc *n′a*, which measures this angle, may therefore be regarded as a right line. But this arc being described with a radius *ma*, if we assume *mb*=*ma* (*Fig.* 153), the angle *bn′a* being an angle in a semicircle, will be equal to a right angle, and may be considered equal to the angle *mn′a*. For, since the angle *n′ma* is infinitely small, the angle *mn′b* must be so likewise, and the angles *bn′a*, *mn′a*, will therefore differ by an infinitely small quantity. Thus, the triangles *mn′a* and *n′la* (*Fig.* 152) being right-angled and having a common angle *a*, will be similar, and we shall therefore have the proportion

$$ma : n'a :: n'a : la.$$

But *n′a* being infinitely small with respect to *ma*, *la* must be infinitely small with respect to *n′a*; and since *n′a* is an infinitely small quantity of the first order, *la* will be one of the second order. Hence, the quantity *la* may be neglected, and *mn′* may be regarded as equal to *ml*; thus we shall have

$$mn' = mm' + m'l.$$

In a similar manner may it be proved that if with the point *n′*

as a centre, and radius $n'm$, we describe the arc ma', we shall obtain

$$mn'=nn'+nh,$$

and by placing these values of mn' equal to each other, we find

$$mm'+m'l=nn'+nh;$$

but the right line mm' being supposed inextensible, it must preserve its length invariable in its new position; hence, $mm'=nn'$; and by suppressing these equal terms in the preceding equation, we obtain

$$m'l=nh.$$

Again, the lines mm' and nn' form with each other an infinitely small angle; for, if they intersect at a point o (*Fig.* 154), we shall have a triangle $m'on'$, two of whose sides are of finite extent, the third side $m'n'$ being infinitely small; thus, the angle o will likewise be infinitely small. It results from the preceding remarks, that if the perpendicular nk be demitted on the side mm' (*Fig.* 152) we shall have

$$nh=mk;$$

and by substituting this value of nh in the preceding equation, we find

$$m'l=mk,$$

which proves that the projections mk and $m'l$ of the virtual velocities mn and $m'n'$ of the points m and m' are equal to each other.

367. Let us now suppose that the point m (*Fig.* 155) is transported to n, and that the extremity m' falls at n' below mm'. It may be proved as in the former case, that the angle o is infinitely small, and consequently that the projections ol and oh may be regarded as equal to on' and on; whence,

$$on'=om'+m'l, \quad on=om-mh;$$

by the addition of these equations, we obtain

$$on'+on=om'+om+m'l-mh;$$

or,

$$nn'=mm'+m'l-mh;$$

but nn' and mm' are equal to each other, and therefore

$$m'l=mh,$$

which proves that the projections of the virtual velocities are still equal.

368. In this demonstration it has been supposed that the derangement of the system is such as to preserve the lines mm' and nn' in the same plane. This restriction is however entirely unnecessary. For, if we suppose that mm' and nn' are not contained in the same plane, we can draw through the points n and n' (*Fig.* 152) planes perpendicular to the line mm', intersecting this line at the points k and l, the projections of n and n'. Then, if a line be drawn through any point of mm' parallel to nn', and terminated by the perpendicular planes, such line will evidently be equal to nn', and its extremities will likewise be projected on the line mm', at the same points k and l. Hence, if the property be true for the parallel line which intersects mm', it will likewise be true for the line nn'.

369. It should be observed, that in each of these cases, the projections will be affected with contrary signs, one falling upon the line mm', the other upon its prolongation.

This appears from an inspection of the figures 152 and 155, and it likewise results from the consideration that if the two projections fell upon the line or upon its prolongations, the length of nn' would necessarily be greater or less than that of mm', which by hypothesis, is impossible.

370. It follows from the preceding remarks, that if we suppose two equal and opposite forces to act in the direction of the line mm' on the points m and m', and denote by v and v' the projections of the virtual velocities mn, and $m'n'$ on the line of direction of the forces, we shall have

$$v=-v';$$

and consequently, that if we represent by (mm') each of these equal forces, we shall obtain

$$(mm')v+(mm')v'=0;$$

which proves that the forces represented by (mm') being applied at the extremities of the right line, and being regarded as sustaining those points in equilibrio, the sum of the moments of the virtual velocities of these points will be equal to zero.

371. By the aid of this proposition it will be easy to

establish the principle of virtual velocities in the case of any number of forces applied to different points. For, let P, P′, P″, &c. (*Fig.* 156), be several forces applied to the points m, m', m'', &c. If we regard these points as firmly connected by inflexible right lines, these lines may be considered as the directions of equal and opposite forces acting on the points m, m', m'', &c., and if we denote these forces by (mm'), $(m'm'')$, &c., the equilibrium will be maintained

at the point m, by the forces (mm'), (mm''), (mm'''), and P,
at the point m', by the forces $(m'm)$, $(m'm'')$, $(m'm''')$, and P′,
at the point m'', by the forces $(m''m)$, $(m''m')$, $(m''m''')$ and P″,
at the point m''', by the forces $(m'''m)$, $(m'''m')$, $(m'''m'')$, and P‴,
&c. &c. &c.

Since the equilibrium subsists at each of these points, the equation of the moments obtained in Art. 365, will manifestly be satisfied. Let the following notation then be adopted, viz: v=projection of the virtual velocity of one of the points m, m', m'', &c., the point to which this velocity refers being designated by the manner in which v is written in the expression for its moment; thus, $v(mm')$ represents that v in this moment applies to the point m, while $v(m'm)$ denotes that v applies to m'.

The character v will thus represent quantities which may be equal or unequal, according as the projections of the virtual velocities fall upon the same or upon different lines.

372. Having adopted this notation, the equations of the moments as given by Art. 365, may be expressed as follows:

for the point m, $Pp+v(mm')+v(mm'')+v(mm''')=0$,
for the point m', $P'p'+v(m'm)+v(m'm'')+v(m'm''')=0$,
for the point m'', $P''p''+v(m''m)+v(m''m')+v(m''m''')=0$,
for the point m''', $P'''p'''+v(m'''m)+v(m'''m')+v(m'''m'')=0$.

The sum of these four equations being taken, we remark that the moments appertaining to the same right line mutually destroy each other; thus, the term $v(mm')$ will cancel the term $v(m'm)$, &c., and by continuing the process, the sum will be reduced to

$$Pp+P'p'+P''p''+P'''p'''=0.$$

The same demonstration is evidently applicable to a greater number of forces.

373. As an example of the manner in which the conditions of equilibrium in any machine may be inferred from the principle of virtual velocities, we will suppose the relation between the power and resistance in the lever to be unknown. The forces exerted upon the lever are the power P, the resistance P′, and the reaction of the point of support. If a slight motion be communicated to the lever, causing it to turn about its fulcrum, this fulcrum will remain immoveable, and the moment of the reaction exerted by this point will therefore be equal to zero. Hence, the principle of virtual velocities will give

$$\mathrm{P}p+\mathrm{P}'p'=0;$$

or,

$$\mathrm{P}p=-\mathrm{P}'p' \; \ldots\ldots \; (146).$$

This being premised, let the values of the quantities p and p' be now determined. Let C represent the fulcrum of a lever mm' (*Fig.* 157), which being slightly removed from its position of equilibrium has assumed the position nn'; the angles at C being equal to each other, the arcs mn, $m'n'$ will be proportional to the radii with which they are described, and we shall therefore have

$$mn : m'n' :: \mathrm{C}m : \mathrm{C}m' \; \ldots\ldots \; (147).$$

But if through the points n and n' perpendiculars be drawn to the directions of the forces P and P′, we shall have

$$mr=-p, \quad m'r'=p',$$

the negative sign being prefixed to p, because it falls on the prolongation of the force P. The arcs being regarded as indefinitely small right lines, the right-angled triangles mrn, $m'r'n'$ will be similar; for the isosceles triangles $m\mathrm{C}n$, $m'\mathrm{C}n'$ give

$$\text{angle } nm\mathrm{C}=\text{angle } n'm'\mathrm{C}:$$

and by subtracting these equal angles from the right angles $rm\mathrm{C}$, $r'm'\mathrm{C}$, there will remain

$$\text{angle } rmn=\text{angle } r'm'n'.$$

Thus, the triangles rmn, $r'm'n'$ will be similar, and will give the proportion

$$mn : m'n' :: mr : m'r';$$

or,

$$mn : m'n' :: -p : p';$$

and therefore the proportion (147) may be converted into

$$Cm : Cm' :: -p : p'.$$

But the equation (146) which expresses the principle of virtual velocities gives rise to the proportion

$$P' : P :: -p : p';$$

whence, by the equality of ratios,

$$Cm : Cm' :: P' : P,$$

or the forces are in the inverse ratio of the arms of the lever.

Of the Position of the Centre of Gravity of a System when in Equilibrio.

374. Let m, m', m'' &c., be the centres of gravity of different bodies which are connected together in an invariable manner; let perpendiculars z, z', z'', &c., be demitted from these points on the plane of xy, supposed to be horizontal; the weights P, P′, P″, &c., of the several bodies, which may be regarded as suspended from the points m, m', m'', &c., will act along the directions of these perpendiculars. If $z_{\prime}$ denote the coordinate of the centre of gravity of the whole system, we shall have (Art. 166)

$$z_{\prime}=\frac{Pz+P'z'+P''z''+\&c.}{P+P'+P''+\&c.}.$$

When the system of bodies changes its position, the ordinate z becoming $z+h$, or $z-h$, the increment of z will affect the values of z', z'', z''', &c., since the points m, m', m'', &c., being connected in an invariable manner, the value of z cannot change without the values of z', z'', &c., undergoing a corresponding alteration. Although we are generally unacquainted with the law of dependence which exists between the positions of the different bodies composing the system, the preceding equation may nevertheless be written under the form

$$z_{\prime}=\frac{Pz+P'\varphi z+P''Fz+\&c.}{P+P'+P''+\&c.},$$

in which φ, F, &c. denote certain indeterminate functions.

If the value of $z_{\prime}$ be a maximum or a minimum, the differential of the second member will be equal to zero, hence

$$Pdz+P'd\varphi z+P''dFz+\&c.=0;$$

or,

$$Pdz+P'dz'+P''dz''+\&c.=0.$$

But this equation is necessarily satisfied when the system receives an infinitely small derangement from its position of equilibrium. For, when the centres of gravity m, m', m'', &c., change their positions and are transferred to n, n', n'', &c., the paths described will be the lines mn, $m'n'$, $m''n''$, &c. If, therefore, these paths be projected on the primitive directions z, z', z'', &c., of the forces P, P′, P″, &c., we shall obtain the values of the projections of the virtual velocities. Thus, mh (*Fig.* 158) the projection of mn upon the co-ordinate z, is equal to nk, the increment which the value of z has received in consequence of the derangement sustained by the system: the sign of this increment may be either positive or negative. We shall therefore have, without reference to the signs, $p=dz$; and by applying the same considerations to the other co-ordinates, it appears that the differential $Pdz+P'dz'+P''dz''+\&c.$, will represent the same quantity as the expression $Pp+P'p'+P''p''+\&c.$; and since the latter quantity becomes equal to zero when the system is in equilibrio, according to the principle of virtual velocities, we must likewise have

$$Pdz+P'dz'+P''dz''+\&c.=0;$$

hence, $dz_{\prime}=0$, which proves that the centre of gravity is *in general* situated at the highest or lowest point, when the system is in a state of equilibrium. But this proposition will not always be true, since $dz_{\prime}=0$ will not always indicate the existence of a maximum or minimum.

375. The converse of this proposition is always true, viz: *If the centre of gravity of the system be situated at the highest or lowest point, the system will necessarily be in equilibrio;* for, $dz_{\prime}$ will then be equal to zero, and the sum of the moments of the virtual velocities will also be equal to zero.

PART SECOND.

DYNAMICS.

OF THE LAW OF INERTIA.

376. DYNAMICS has been defined to be that part of Mechanics which treats of the laws of motion of solid bodies. We shall, in the first place, establish as a principle the general law of nature, that every body will continue in the state of rest or motion in which it may be placed, unless it be acted upon by some external force. This indifference of matter to a state of motion or rest is called *inertia.* It is a consequence of this principle of inertia that one body when struck by another, exerts an effort of resistance to the impulsion, whilst acquiring a portion of the motion of the striking body. By this same principle, a body having received an impulse, must move uniformly in a right line, if not opposed by any obstacle: for there can be no reason why the body should deviate to one side rather than to the other, nor that its motion should be accelerated rather than retarded. It is true, that the nature of the force being unknown to us, we cannot foresee whether its effect will be such as to preserve the motion of the body invariable: thus, the law of inertia should be regarded as a simple result of experience and analogy.

If we do not perceive the motions of bodies to continue unchanged, it is merely because these motions are constantly affected by the resistance of media, by the action of gravity, or by other similar causes. The most simple kind of motion which can be conceived is that which takes place uniformly, and in a right line.

Of Uniform Rectilinear Motion.

377. A body is said to have a *uniform motion* when it passes over equal spaces in successive equal portions of time: thus, if V denote the space which it describes in a unit of time, it will have described a space 2V at the end of two units of time, 3V at the end of three units of time, &c. Consequently, if we represent by t the number of units of time necessary for the body to describe a space s, this space will be equal to $t\times V$; we shall thus have

$$s=Vt.$$

Such is the equation of uniform motion. The coefficient V, or the space passed over in a unit of time, is called the *velocity*, and it evidently expresses the rate of a body's motion. For, if a body M move n times as rapidly as another M′, the space V described by the first in a unit of time, will obviously be n times greater than the space V′, described by the second in the same time.

378. For the purpose of comparing the circumstances of motion of two bodies which depart at the same instant from a point A, with velocities represented by V′ and V″, we will denote by s' and s'' the respective spaces passed over by these bodies, at the expiration of the times t' and t'': we shall then have

$$s'=V't', \quad s''=V''t'';$$

whence we deduce

$$\frac{s'}{s''}=\frac{V't'}{V''t''};$$

which proves that *the spaces passed over are proportional to the products of the times and velocities.* When the times are equal, this equation reduces to

$$\frac{s'}{s''}=\frac{V'}{V''},$$

and the spaces described are then proportional to the velocities.

379. The body may have already passed over a space S, previous to the instant from which the time t is reckoned:

we shall then have the more general equation of uniform motion

$$s=\mathrm{S}+\mathrm{V}t,$$

in which s represents the distance of the body from the origin of spaces. The quantity S is called the initial space, and evidently represents the distance of the body from the origin, at the commencement of the time t.

380. By the aid of this equation we can readily solve all the problems of uniform rectilinear motion.

For example, if the distance of a body from the origin of spaces at the end of the time t', be supposed equal to s'; and if this distance become s'' at the end of the time t'', we can thence determine the velocity V, and the initial space; for we shall have the equations

$$s'=\mathrm{S}+\mathrm{V}t', \quad s''=\mathrm{S}+\mathrm{V}t'';$$

from which we obtain

$$\mathrm{V}=\frac{s''-s'}{t''-t'}, \quad \mathrm{S}=\frac{s't''-s''t'}{t''-t'}.$$

381. As a second example, let it be required to determine the time of meeting of two bodies M′ and M (*Fig.* 159), which depart at the same instant from the two points A and B, having the respective velocities V′ and V. Let C be their point of meeting: the spaces actually passed over by the two bodies will be

$$\mathrm{BC}=\mathrm{V}t, \text{ and } \mathrm{AC}=\mathrm{V}'t.$$

If we denote by b the distance AB between the bodies at the commencement of the motion, and reckon their distances at the end of the time t from the point A as an origin, we shall have the equations

$$s=b+\mathrm{V}t, \quad s'=\mathrm{V}'t.$$

Each of the spaces s and s' will then be represented by the line AC; and by placing the second members of the above equations equal to each other, we deduce

$$t=\frac{b}{\mathrm{V}'-\mathrm{V}}.$$

382. Since the space s constantly varies with the time t,

we can differentiate the equation $s=S+Vt$ with reference to these two variables, and we shall thus obtain

$$\frac{ds}{dt}=V.$$

Hence it appears, that in uniform motion the velocity is the differential coefficient of the space, regarded as a function of the time: it will presently appear that the same is true in varied motion.

Of Varied Motion.

383. When the motion of a body is such that it passes over unequal spaces in equal successive portions of time, the body is said to have a *varied motion*. This kind of motion cannot be produced by the action of a single force of impulsion, since by the law of inertia the velocity imparted by a single impulse should constantly remain unchanged; and hence the motion would continue uniform: whereas, we have in the present instance supposed it variable. It therefore becomes necessary to suppose that the body, having received the first impulsion, is subsequently subjected to the action of a second impulse, a third, &c., which, by constantly changing its velocity, produce a variable motion. If the force acts without intermission, the impulses will be communicated at intervals which are indefinitely small, and the force is then called an *incessant force*. If the force tends to increase the velocity of the body, it is called an *accelerating force*, and when it tends to diminish the velocity, a *retarding force*.

384. The velocity of the body being supposed constantly variable, we can only estimate its value at any particular point of the path described, by supposing it to become constant at this point. Thus, to measure the velocity of a body which has arrived at B (*Fig.* 159), at the end of the time t, we suppose the action of the incessant force to be suddenly arrested, and the body will then move uniformly with the velocity which it has acquired at the point B. The space BC described in a unit of time, with this uniform motion, is the measure of the velocity at the point B.

385. The *second* is usually adopted as the unit of time. Hence, the velocity of the body at the expiration of the time t will be the space which this body would describe in the second which succeeds the time t, if, at the end of the time t, the incessant force should cease to communicate new impulses to the body.

386. To determine the analytical expression for the velocity, we will suppose the body to have arrived at the point B, at the expiration of the time t; the space AB which it has already passed over being dependent on the length of time which has elapsed, the former will evidently be a function of the latter. Thus, we may regard the space s as the ordinate of a curve whose absciss is equal to t; consequently, when t becomes $t+dt$, s will become $s+ds$; hence, the space passed over in the time dt will be represented by ds. This being premised, let it be supposed that when the body has arrived at the point B, the incessant force ceases to act; the body will assume a uniform motion with the velocity acquired at the point B, and will describe in the instant dt succeeding the time t, the indefinitely small space ds: in the next succeeding instant dt it will describe a second space ds, and the same will continue until the body has described a space BC, which will correspond to the unit of time. This space BC will therefore contain ds as many times as dt is contained in unity; but $\frac{1}{dt}$ will express the number of times which the unit of time contains the quantity dt; hence, the space BC will be expressed by $ds\times\frac{1}{dt}$, or by $\frac{ds}{dt}$, since the differential is taken with reference to the variable t; but the space BC represents the quantity v; we shall therefore have, for the expression of the velocity in varied motion

$$v=\frac{ds}{dt}.$$

387. It may also be observed that the space passed over, after the expiration of the time t, will be (*Fig.* 159),

$Bb=ds$ at the end of the time dt,
$Bb'=2ds$ at the end of the time $2dt$,

$Bb''=3ds$ at the end of the time $3dt$.

. . . .

$BC=nds$ at the end of the time $n.dt$.

And since the time elapsed during the passage of the body from B to C is by hypothesis equal to unity, we may suppose $ndt=1$; whence $n=\frac{1}{dt}$. This value being substituted in the expression $nds=v$, the space described in a unit of time, we shall obtain, as above,

$$v=\frac{ds}{dt} \quad \ldots. (148).$$

388. Before investigating the expression for the value of the incessant force, it will be necessary to discover the relation which exists between the force and the velocity.

If a force F be supposed to communicate a velocity v to any body, a force n times as great will communicate to the body a velocity equal to nv. The truth of this proposition might well be questioned, since the nature of forces being entirely unknown, we cannot affirm that a double force will necessarily produce a double velocity; or, in general, that a single force equal to the sum of two others, will necessarily produce a velocity equal to the sum of the velocities which the two forces would separately produce. But the fact being confirmed by universal experience, we adopt it as a principle. Thus, by supposing different forces applied to the same body or material point, their relative intensities can be estimated by comparing the velocities which they would severally communicate.

The proper measure of an incessant force will be the velocity which it can generate in a given time; but the intensity of the force being constantly variable, we must suppose the force to become constant at the instant when we wish to estimate its value, and the measure of the force will then be the velocity generated in the unit of time succeeding this instant. The velocity communicated by this incessant force during the unit of time, when it is supposed to retain a constant value, will obviously be unequal to that which would have been communicated by the variable incessant force, in the same time.

389. The preceding remarks indicate the method of meas-

uring the incessant force; since they determine the ratio in which the intensity of the force varies in different times.

If, for example, at the expiration of the times t and t', the incessant force, having become constant, can generate in a second of time velocities represented by the numbers 60 and 20, we infer that the intensity of the force at the end of the time t is triple its intensity at the end of the time t'.

390. To deduce from the above definition the analytical expression for the incessant force, let v represent the velocity acquired by the body at the end of the time t; then, at the expiration of the time $t+dt$, the velocity will become $v+dv$; consequently, dv will be the velocity communicated during the time dt; but if at the end of the time t the intensity of the force be supposed to become constant, there will be communicated to the body in the instant dt which succeeds the time t, a velocity represented by dv; and the same effect will be repeated during any number of succeeding instants; so that the velocities communicated after the expiration of the time t, in the instants dt, $2dt$, $3dt$, &c., will be expressed by dv, $2dv$, $3dv$, &c.: and consequently, the velocity communicated in the unit of time which succeeds the time t, will be equal to dv repeated as many times as dt is contained in unity. This number being expressed by $\frac{1}{dt}$, it follows that $\frac{1}{dt}\times dv$, or $\frac{dv}{dt}$, will express the effect of the force, or the velocity generated in a unit of time. If, therefore, we denote this force by φ, we shall obtain for the second equation of varied motion,

$$\varphi=\frac{dv}{dt} \quad \ldots\ldots (149).$$

The character φ will hereafter be used to designate the intensity of the force; the force being represented by the effect which it produces.

391. From the preceding equation we obtain

$$\varphi dt=dv;$$

thus, if the incessant force be given, the increment to the velocity in the time dt can be readily calculated.

392. By eliminating dt between the equations (148) and (149), we obtain a third equation of varied motion,

$$\varphi ds=vdv.$$

Of Uniformly Varied Motion.

393. The incessant force imparting at each instant a new impulse to the body, if these impulses are equal in intensity, the body will acquire the same velocity in a unit of time after the expiration of the time t, as it would after a time t'. Let this velocity which is constantly generated in a unit of time, be denoted by g; we shall then have

$$\phi=g.$$

Substituting this value in the equation

$$\phi=\frac{dv}{dt},$$

we shall obtain

$$dv=gdt;$$

and by integrating and denoting by a the constant which will thus be introduced, we find

$$v=a+gt\ .\ .\ .\ .\ .\ (150).^{*}$$

We have likewise obtained for the value of the velocity

$$v=\frac{ds}{dt};$$

hence, if we eliminate v between these two equations, we shall have

$$ds=(a+gt)dt,$$

from which, by integration, we find

$$s=b+at+\tfrac{1}{2}gt^2\ .\ .\ .\ .\ .\ (151),$$

the quantity b being an arbitrary constant.

---

[*] This equation might also have been obtained from the following considerations: Let it be supposed that a body in motion has acquired a velocity a: if it then be solicited by a constant force which communicates to it a velocity g in each second of time, the velocity of the body will become

$a+g$, at the end of one second,
$a+2g$, at the end of two seconds,
$a+3g$, at the end of three seconds,
. . . .
$a+tg$, at the end of t seconds:

thus, if we represent by v the velocity of the body at the expiration of the time t, we shall have

$$v=a+gt.$$

If g be supposed positive in this equation, the motion will be *uniformly accelerated*, but if negative, the motion will be *uniformly retarded*.

394. If we make $t=0$, we find $b=s$; thus, b will represent the initial space, or the distance of the body from the origin, at the instant from which the time is reckoned.

The constant a is equal to the initial velocity of the body, as appears by making $t=0$ in equation (150).

395. When the initial space and initial velocity are each equal to zero, the equations (150) and (151) become

$$v=gt \ldots\ldots (152),$$

$$s=\tfrac{1}{2}gt^2 \ldots\ldots (153),$$

and the body then moves from rest, under the action of the incessant force.

396. Let s and s' represent the spaces described in the times t and t', under the action of a force g; the equation (153) gives

$$s=\tfrac{1}{2}gt^2, \quad \text{and } s'=\tfrac{1}{2}gt'^2 \ldots\ldots (154);$$

whence we obtain the proportion

$$s : s' :: t^2 : t'^2 \ldots\ldots (155).$$

Consequently, *the spaces described by a body in different times, when it moves from rest, being solicited by a constant accelerating force, are proportional to the squares of those times.*

397. The equation (152) gives

$$v=gt, \quad \text{and } v'=gt',$$

whence,

$$v : v' :: t : t',$$

and by comparing this proportion with (155), we have

$$v : v' :: \sqrt{s} : \sqrt{s'}.$$

Hence it appears that *the times elapsed are constantly proportional to the velocities, or to the square roots of the spaces described in those times.*

398. If we make $t=1$, the equation (153) becomes

$$s=\tfrac{1}{2}g.$$

In this case, s represents the space described by the body in the first unit of time, and it appears that this space is equal to one-half the quantity g, which represents the measure of the accelerating force. It has been found, for example, that a body subjected to the action of gravity, would describe in the first second of time, in the latitude of New-York a distance equal to

$$16.0799 \text{ feet, or nearly } 16\tfrac{1}{12} \text{ feet};$$

this value being substituted in the place of s in the preceding equation, we find

$$g=32.1598 \text{ feet, or nearly } = 32\tfrac{1}{6} \text{ feet.}$$

399. The equation (153) will determine the space described in a given time; for example, if $t=6''$, we shall have

$$s=\tfrac{1}{2}gt^2=\tfrac{1}{2}(32\tfrac{1}{6}^{\text{ft.}})\times 36=579 \text{ feet};$$

thus a body being elevated to the height of 579 feet, would require six seconds to fall to the surface of the earth.

400. The velocity acquired by this body, when it has reached the surface, may be determined from equation (150), in which we make

$$a=0, \quad g=32\tfrac{1}{6} \text{ feet}, \quad t=6''.$$

We thus find

$$v=32\tfrac{1}{6}^{\text{ft.}}\times 6=193^{\text{ft.}}.$$

401. If it be required to determine the height from which a body must fall to acquire a given velocity, we eliminate t between the equations

$$s=\tfrac{1}{2}gt^2, \quad v=gt;$$

and we thus obtain

$$v=\sqrt{(2gs)} \; \ldots\ldots \; (156).$$

Let it be supposed, for example, that we wish to determine the space through which a body would fall in acquiring a velocity of 386 feet per second; we shall have

$$386^{\text{ft.}}=\sqrt{(2\times 32\tfrac{1}{6}^{\text{ft.}}\times s)}=\sqrt{(64\tfrac{1}{3}^{\text{ft.}}\times s)};$$

whence,

$$s=\frac{(386^{\text{ft.}})^2}{64\tfrac{1}{3}^{\text{ft.}}}=\frac{148996^{\text{ft.}}}{64\tfrac{1}{3}}=2316^{\text{ft.}}.$$

The velocity acquired in falling through a given height is called *the velocity due to that height.*

402. To determine the time in which a body will fall

through a given height s, we employ the equation (153), which gives

$$t=\sqrt{\left(\frac{2s}{g}\right)}.$$

403. The general equations of variable motion

$$\frac{ds}{dt}=v, \quad \frac{dv}{dt}=\phi \; \ldots\ldots \; (157),$$

will now be applied to the investigation of the circumstances of varied motion under different hypotheses. This investigation is reduced to the determination of the relations which exist between the time elapsed, the space described, and the velocity acquired, since, if the two latter can be expressed in functions of the time, we shall be able to discover the place of the body, and the velocity with which it moves at any given instant. Thus, the circumstances of motion will be entirely known.

Of the Motion of a Body projected Vertically upward.

404. When the action of gravity is alone exerted on a body, we have the relation

$$v=gt,$$

in which v expresses the velocity at the end of the time t: but if we suppose the body instead of moving from rest, to be projected vertically in a direction opposed to that of gravity, with a velocity a, this velocity will have been diminished at the end of the time t, by a quantity equal to the velocity which gravity could impart in the same time; consequently, the velocity of the body at the expiration of the time t will be represented by $a-gt$; and if we represent this velocity by v, we shall have

$$v=a-gt \; \ldots\ldots \; (158):$$

substituting for v, its value $\frac{ds}{dt}$, we find, by integration,

$$s=at-\tfrac{1}{2}gt^2.$$

The initial space being supposed equal to zero, no constant has been added in this integration.

This equation being placed under the form

$$s=(a-\tfrac{1}{2}gt)t;$$

if we substitute for t its value deduced from equation (158), we shall obtain

$$s=\frac{a+v}{2}\times\frac{a-v}{g};$$

or,

$$s=\frac{a^2-v^2}{2g}\ .\ .\ .\ .\ .\ (159).$$

405. The equations (158) and (159) make known all the circumstances of the motion under consideration. Thus, the equation (158) indicates that the velocity constantly decreases as the time increases; and the equation (159) proves that the velocity decreases as the space described becomes greater: hence, the velocity constantly becomes less as the body rises. When this velocity becomes equal to zero, the body has attained its greatest elevation: if we denote this elevation by h, the equation (159) will give, by making $v=0$,

$$h=\frac{a^2}{2g}\ .\ .\ .\ .\ .\ (160).$$

To determine the time corresponding to this elevation, we make $v=0$, in equation (158), and thence deduce

$$t=\frac{a}{g}\ .\ .\ .\ .\ .\ (161).$$

The velocity due to the height h is found by making $h=s$ in the formula

$$v=\sqrt{(2gs)}=\sqrt{(2gh)};$$

and by substituting the value of h deduced from equation (160), we obtain

$$v=\sqrt{\left(2g\times\frac{a^2}{2g}\right)}=a;$$

hence, the body acquires the same velocity in descending, that it lost in ascending.

406. Let it be required to determine the greatest height to which a body will rise when projected vertically upward with a velocity of 100 feet per second: we shall find from equations (160) and (161), that the greatest height is $155\frac{44}{193}$ feet,

and that the time of rising or falling is equal to 3¼ seconds, nearly.

407. The preceding equations may likewise be applied to the case in which the body is projected downward, by simply changing the sign of the quantity g; we shall thus have an expression for the velocity v,

$$v=a+gt.$$

Of the Vertical Motion of a Body when acted upon by the Force of Gravity considered as variable.

408. Gravity is a force whose intensity varies at different distances from the earth's centre. The law of this variation has been discovered to be that of the inverse ratio of the square of the distance; that is to say, that at distances from the centre of the earth represented by 2, 3, 4, &c., it becomes $\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}$, &c., of its value at the distance unity. Thus, although a body falls through a distance of $16\frac{1}{12}$ feet in the first second of time at the surface of the earth, it would fall through a much less space in the same time if the distance of the body from the centre were greatly increased.

409. Let a body be supposed to depart from rest at the point A (*Fig.* 160), and let it be required to ascertain the velocity of the body when it has reached the point B. Denote by g the intensity of the force of gravity at M, the surface of the earth, and by φ its intensity at the point B; by r the radius of the earth CM, and by x the distance from B to C: for the purpose of simplifying the calculation, let the known distance AC be assumed as the linear unit. The force being supposed to vary in the inverse ratio of the square of the distance from the earth's centre, we shall have

$$g:\varphi::x^2:r^2;$$

whence,

$$\varphi=\frac{gr^2}{x^2}.$$

But the general expression for the incessant force being

$$\varphi=\frac{dv}{dt},$$

we shall obtain, by placing these values of φ equal to each other,

$$\frac{dv}{dt}=\frac{gr^2}{x^2} \;\ldots\ldots\; (162).$$

Again, the velocity being equal to the differential of the space divided by the differential of the time, it will be represented by

$$v=\frac{d(1-x)}{dt},$$

or by its equal

$$v=-\frac{dx}{dt} \;\ldots\ldots\; (163).$$

Multiplying the terms of this equation by the corresponding terms of equation (162), we find

$$vdv=-gr^2\frac{dx}{x^2},$$

and by integration,

$$\frac{v^2}{2}=\frac{gr^2}{x}+\mathrm{C}.$$

The constant may be determined from the consideration that when $x=\mathrm{AC}=1$, $v=0$; hence,

$$\mathrm{C}=-gr^2.$$

This value substituted in the preceding equation gives

$$\frac{v^2}{2}=gr^2\left(\frac{1}{x}-1\right) \;\ldots\ldots\; (164).$$

This equation determines the value of the velocity at any given point of the line AC.

410. To determine the time employed by the body in describing the space AB, we eliminate v, between this equation and the equation (163), and we thus obtain

$$\frac{dx^2}{2dt^2}=gr^2\left(\frac{1}{x}-1\right);$$

whence,

$$dt^2=\frac{1}{2gr^2}\times\frac{dx^2}{\frac{1}{x}-1};$$

and consequently,

$$dt=\pm\frac{1}{r}\sqrt{\frac{1}{2g}}\times\frac{dx}{\sqrt{\left(\frac{1}{x}-1\right)}};$$

by the integration of this equation we shall obtain

$$t=\pm\frac{1}{r}\sqrt{\frac{1}{2g}}\int\frac{dx}{\sqrt{\left(\frac{1}{x}-1\right)}}\ldots\ldots(165).$$

To effect the integration which is here only indicated, we reduce the fraction to a simpler form, thus

$$\int\frac{dx}{\sqrt{\left(\frac{1}{x}-1\right)}}=\int\frac{dx}{\sqrt{\left(\frac{1-x}{x}\right)}}=\int\frac{dx\sqrt{x}}{\sqrt{(1-x)}}\ldots\ldots(166).$$

The radical in the denominator may be caused to disappear, by making

$$1-x=z^2.$$

We deduce from this equation,

$$dx=-2zdz,\quad \sqrt{(1-x)}=z,\quad \sqrt{x}=\sqrt{(1-z^2)}.$$

These values substituted in the preceding formula give

$$\int\frac{dx\sqrt{x}}{\sqrt{(1-x)}}=-2\int dz\,.\sqrt{(1-z^2)}\ldots\ldots(167).$$

Integrating by parts, we find

$$\int dz\sqrt{(1-z^2)}=z\sqrt{(1-z^2)}+\int\frac{z^2dz}{\sqrt{(1-z^2)}}.$$

But we likewise have the identical equation

$$\int dz\sqrt{(1-z^2)}=\int\frac{dz}{\sqrt{(1-z^2)}}-\int\frac{z^2dz}{\sqrt{(1-z^2)}}.$$

Adding these equations, and dividing by 2, we obtain

$$\int dz\sqrt{(1-z^2)}=\tfrac{1}{2}z\sqrt{(1-z^2)}+\tfrac{1}{2}\int\frac{dz}{\sqrt{(1-z^2)}}$$

$$=\tfrac{1}{2}z\sqrt{(1-z^2)}+\tfrac{1}{2}\text{ arc }(\sin=z);$$

consequently,

$$-2\int dz\sqrt{(1-z^2)}=-z\sqrt{(1-z^2)}-\text{arc}(\sin=z),$$

and by substituting this value in the equation (167), we shall obtain the integral of (166); hence, the equation (165) will become

$$t=\pm\frac{1}{r}\sqrt{\frac{1}{2g}}[z\sqrt{(1-z^2)}+\text{ arc }(\sin=z)]\ldots\ldots(168).$$

The constant will be equal to zero, since $x=1$, when $t=0$; and therefore, $z=\sqrt{(1-x)}=0$; this supposition causes the second member of the equation to vanish. Moreover, the time being essentially positive, we use only the inferior sign in the preceding equation; and by observing that $z^2 = 1-x =$ the distance AB which the body has described, we shall have, by representing this distance by s,

$$t=\frac{1}{r}\sqrt{\frac{1}{2g}}[\sqrt{s}\sqrt{(1-s)}+\text{arc }(\sin=\sqrt{s})].$$

411. This last equation is much simplified by supposing the distances AB and AM to be exceedingly small when compared with the distances AC and MC; for, the quantity $\sqrt{(1-s)}$ may then, without sensible error, be supposed equal to unity; and the arc $(\sin=\sqrt{s})$ may likewise be considered as equal to its sine; hence, by changing r into unity, the preceding expression will reduce to

$$t=\sqrt{\frac{1}{2g}}\times 2\sqrt{s};$$

and from this we deduce the relation

$$s=\tfrac{1}{2}gt^2;$$

or, the motion is then similar to that which would take place if the intensity of the force remained invariable.

Of the Vertical Motion of a Body in a resisting Medium.

412. It has been ascertained that a body when moving in a fluid experiences a resistance which is proportional to the square of the velocity. Thus, by calling m the intensity of this resistance when the velocity of the body is represented by unity, the resistance will be expressed by mv^2, when the body has acquired a velocity v.

413. This force being opposed to that of gravity when the body descends, we shall have, by supposing the intensity of gravity constant,

$$\varphi=g-mv^2;$$

and by substituting for φ its general value $\frac{dv}{dt}$, we obtain

$$\frac{dv}{dt}=g-mv^2;$$

whence,

$$dt=\frac{dv}{g-mv^2}\;.\;.\;.\;.\;.\;(169).$$

To integrate this equation, we decompose the denominator into factors, and thus have

$$g-mv^2=(\sqrt{g}+v\sqrt{m})(\sqrt{g}-v\sqrt{m}).$$

If we then suppose, according to the method of rational fractions,

$$\frac{dv}{g-mv^2}=dv\left(\frac{\mathrm{A}}{\sqrt{g}+v\sqrt{m}}+\frac{\mathrm{B}}{\sqrt{g}-v\sqrt{m}}\right)\;.\;.\;.\;.\;.\;(170),$$

we shall find, by reducing the terms of the second member to a common denominator, and placing the coefficients of the like powers of v equal to each other,

$$\mathrm{A}=\mathrm{B}=\frac{1}{2\sqrt{g}};$$

these values substituted in the equation (170), give

$$\frac{dv}{g-mv^2}=\frac{1}{2\sqrt{g}}\left(\frac{dv}{\sqrt{g}+v\sqrt{m}}+\frac{dv}{\sqrt{g}-v\sqrt{m}}\right).$$

Multiplying and dividing the second member of this equation by $\sqrt{m}$, we shall obtain a value, which substituted in equation (169) will reduce it to

$$dt=\frac{1}{2\sqrt{m}\sqrt{g}}\left(\frac{dv\sqrt{m}}{\sqrt{g}+v\sqrt{m}}+\frac{dv\sqrt{m}}{\sqrt{g}-v\sqrt{m}}\right);$$

and by integration, we obtain

$$t=\frac{1}{2\sqrt{(mg)}}\left(\log(\sqrt{g}+v\sqrt{m})-\log(\sqrt{g}-v\sqrt{m})\right)+\mathrm{C};$$

or,

$$t=\frac{1}{2\sqrt{(mg)}}\log\frac{\sqrt{g}+v\sqrt{m}}{\sqrt{g}-v\sqrt{m}}\;.\;.\;.\;.\;.\;(171).$$

The constant may be suppressed, since when $t=0$, $v=0$.

414. If the two members of the equation (171) be multiplied by $2\sqrt{mg}$, and the first member by the logarithm of the base e of the Naperian system, which is equal to unity, we shall have

$$2t\sqrt{(mg)}\,.\log e=\log\frac{\sqrt{g}+v\sqrt{m}}{\sqrt{g}-v\sqrt{m}},$$

or,

$$\log e^{2t\sqrt{(mg)}}=\log\frac{\sqrt{g}+v\sqrt{m}}{\sqrt{g}-v\sqrt{m}};$$

and by passing to the numbers, we have

$$e^{2t\sqrt{(mg)}}=\frac{\sqrt{g}+v\sqrt{m}}{\sqrt{g}-v\sqrt{m}}.$$

415. This equation being written under the form

$$\frac{1}{e^{2t\sqrt{(mg)}}}=\frac{\sqrt{g}-v\sqrt{m}}{\sqrt{g}+v\sqrt{m}}\;.\;.\;.\;.\;.\;(172),$$

it is obvious that if t be supposed to increase indefinitely, the value of the first member will approach to zero; and consequently, when t becomes infinite, we shall have

$$\sqrt{g}-v\sqrt{m}=0\;.\;.\;.\;.\;.\;(173)$$

From this equation we deduce

$$v=\frac{\sqrt{g}}{\sqrt{m}}=\text{a constant quantity.}$$

Hence we conclude, that as the time increases the velocity becomes more nearly constant.

416. To determine the space described in functions of the velocity, we multiply the corresponding terms of the equations

$$\frac{dv}{dt}=g-mv^2,\quad v=\frac{ds}{dt};$$

and we thus find

$$vdv=(g-mv^2)ds;$$

whence,

$$ds=\frac{vdv}{g-mv^2}\;.\;.\;.\;.\;.\;(174).$$

This equation may be rendered integrable by making

$$g-mv^2=z.$$

For, we obtain by differentiation,

$$vdv=-\frac{dz}{2m};$$

and these values substituted in equation (174), transform it into

$$ds=-\frac{dz}{2mz},$$

the integral of which is

$$s=-\frac{1}{2m}\log z+\mathrm{C};$$

or, by replacing z by its value $g-mv^2$, we have

$$s=-\frac{1}{2m}\log(g-mv^2)+\mathrm{C}.$$

The constant C may be determined by making $s=0$, and $v=0$; whence,

$$\mathrm{C}=\frac{1}{2m}\log g;$$

which value being substituted in the preceding equation, gives

$$s=\frac{1}{2m}[\log g-\log(g-mv^2)];$$

or, finally,

$$s=\frac{1}{2m}\log\left(\frac{g}{g-mv^2}\right).$$

Of the Motions of Bodies upon Inclined Planes.

417. Let a body be situated upon an inclined plane, and let the weight of this body, considered as a vertical force applied at its centre of gravity, be resolved into two components, which shall be respectively parallel and perpendicular to the surface of the plane. The perpendicular force, being supposed to pass through a point of contact, will evidently be destroyed by the resistance of the plane, while the parallel component will cause the centre of gravity to describe a line parallel to the plane. The question will thus be reduced to the consideration of the motion of a material point upon the inclined plane.

418. Let m represent the material point (*Fig.* 161), and g the velocity which gravity can impart in a unit of time: if the force of gravity, represented by the vertical line mB, be resolved into two components mD and mC, respectively parallel and perpendicular to the plane, the latter will be destroyed by the resistance of the plane, and the former will cause the material point to slide along the plane.

But, since forces are proportional to the velocities which they communicate in the same time, if we denote by g' the velocity communicated in a unit of time by the component which acts in the direction of the plane, we shall have

$$mB : mD :: g : g'.$$

The ratio between mB and mD being the same as that between the length and the height of the plane, we shall have, by representing these quantities by h' and h respectively,

$$g : g' :: h' : h;$$

whence,

$$g' = \frac{gh}{h'} \; \ldots\ldots \; (175).$$

419. From this equation it appears, that the velocity g', which is generated in a unit of time by the component of gravity parallel to the plane, is equal to the velocity g, multiplied by the constant ratio $\frac{h}{h'}$; and we therefore conclude that the force which urges the body along the inclined plane differs from the force of gravity only in its intensity. Hence, if we denote by t' the time requisite to describe the entire distance $mA = h'$, the same relations will exist between the quantities g', h', and t', as have been already obtained between g, h, and t, in investigating the circumstances of uniformly varied motion: we shall therefore have

$$h' = \tfrac{1}{2}g't'^2 \; \ldots\ldots \; (176);$$

and the velocity acquired by the body at the point A will be

$$v' = g't';$$

or by eliminating the time t', we shall find

$$v' = \sqrt{(2g'h')}.$$

If in this equation we substitute for g' its value found in equation (175), we shall obtain, after reduction,

$$v' = \sqrt{(2gh)}.$$

The expression for the velocity being independent of the angle mAE, which the inclined plane forms with the horizon, it follows that if several bodies be allowed to descend from the same point m upon different inclined planes mA, mA', mA'', &c. (*Fig*. 162), they will all have acquired the same velocity when they shall have arrived at the same horizontal plane.

420. Although the velocities acquired at the points A and E are equal, the times of descent will be unequal; for, if t and t' represent the times of describing mE and mA, their values will result from the equations

$$t=\sqrt{\frac{2h}{g}},\quad t'=\sqrt{\frac{2h'}{g'}};$$

but we have

$$h<h',$$

$$g>g',\quad \text{or } \frac{1}{g}<\frac{1}{g'};$$

and from these inequalities we deduce

$$\frac{2h}{g}<\frac{2h'}{g'};$$

which proves that the value of t' exceeds that of t.

421. In general, if t' and t'' represent the times of describing two inclined planes h' and h'', having a common altitude h; and if g' and g'' represent the components of gravity respectively parallel to these planes, we shall have

$$t'=\sqrt{\frac{2h'}{g'}},\quad t''=\sqrt{\frac{2h''}{g''}};$$

whence,

$$t':t''::\sqrt{\frac{2h'}{g'}}:\sqrt{\frac{2h''}{g''}};$$

or by replacing g' and g'' by their values (Art. 418), we obtain

$$t':t''::\sqrt{\left(\frac{2h'\times h'}{gh}\right)}:\sqrt{\left(\frac{2h''\times h''}{gh}\right)}::\sqrt{h'^2}:\sqrt{h''^2}::h':h''.$$

Thus, the times of describing different inclined planes having a common altitude will be proportional to the lengths of those planes.

422. The motions of bodies upon inclined planes give rise to a remarkable mechanical property of the circle: it consists in this,—that if the plane of the circle be supposed vertical, the body will require the same time to describe a chord AC (*Fig.* 163), as is necessary to fall through the vertical diam-

eter AB. For, the equation (176) gives, for the time of descent through AC,

$$t'=\sqrt{\frac{2h'}{g'}};$$

and by substituting for g' its value $\frac{gh}{h'}$, this equation will become

$$t'=\sqrt{\frac{2h'^2}{gh}} \ldots\ldots (177).$$

But if the diameter of the circle be denoted by d, we shall have, by the property of the circle,

$$\text{AB} : \text{AC} :: \text{AC} : \text{AD};$$

or,

$$d : h' :: h' : h;$$

and consequently,

$$h'^2=dh.$$

This value substituted in equation (177), gives, after reduction,

$$t'=\sqrt{\frac{2d}{g}}:$$

but this value is precisely the same as that which has been found for the time t, in which the body would fall through the diameter AB: for, the height AB being expressed by d, we shall have

$$d=s=\tfrac{1}{2}gt^2;$$

whence,

$$t=\sqrt{\frac{2d}{g}}.$$

Of Curvilinear Motion.

423. We have hitherto supposed the motion under consideration to be rectilinear; but if it be curvilinear, the space described, and the velocity acquired in a given time, will be insufficient to determine all the circumstances of the motion: it will likewise be necessary to know the nature of the curve described by the body, and the point of this curve at which the body is found at the end of a given time.

424. In the resolution of this problem, we employ the principle of the parallelogram of velocities, which is similar to that of the parallelogram of forces. It may be enunciated as follows: *If two forces* P *and* Q (*Fig.* 164) *communicate, in a unit of time, to a material point m, velocities represented by* mB *and* mC *respectively, the resultant* R *of* P *and* Q *will communicate to the point, in the same time, a velocity* mD, *which will be represented by the diagonal of the parallelogram constructed on the lines* mB *and* mC. The truth of this proposition may be thus established:—Let the force P be represented by the line mB; then, since forces are proportional to the velocities which they communicate in a given time, the force Q will be represented by the line mC. But, by regarding mBDC as the parallelogram of forces, the diagonal mD will represent the resultant of the forces P and Q; and it is required to prove that the velocity resulting from the composition of the two velocities mB and mC is the same as that which is due to the force R. Let x represent the velocity which the force R can communicate to the point m in a unit of time; then, since forces are proportional to the velocities which they generate, we shall have

$$\mathrm{P} : \mathrm{R} :: m\mathrm{B} : x.$$

But from the parallelogram of forces, we deduce

$$\mathrm{P} : \mathrm{R} :: m\mathrm{B} : m\mathrm{D};$$

hence,

$$m\mathrm{B} : m\mathrm{D} :: m\mathrm{B} : x;$$

and therefore,

$$x = m\mathrm{D}.$$

425. In the preceding remarks the forces P, Q, and R have been supposed to act incessantly, communicating new impulses at each successive instant of time. The results obtained will however be equally true if we regard P, Q, and R as impulsive forces which communicate their effects instantaneously, since the velocities imparted by such forces are proportional to the intensities of the forces.

426. The composition of three velocities by the construction of a parallelopiped, results immediately from the preceding principle; for, let P, Q, and R (*Fig.* 165) represent

three forces which communicate the velocities *mp*, *mq*, and *mr* to the material point *m*; let the velocities *mp* and *mq* be compounded into a single velocity *mp'*, which, by the preceding demonstration, will be the same as that communicated by the force P', the resultant of the two forces P and Q: in like manner, the resultant *ms* of the two velocities *mp'* and *mr*, will represent the velocity communicated by the force S, the resultant of the two forces P' and R, or of the three forces P, Q, and R; hence, the diagonal of the parallelopiped constructed on the lines representing the three velocities will represent the velocity communicated by the resultant of the three forces P, Q, and R.

427. We will now examine the circumstances in which a material point will describe a curvilinear path. For this purpose, let the material point *m* (*Fig.* 166), at rest, be supposed to yield to the effect of an impulsion which causes it to describe the right line *m*A in the time *t*, and at the end of this time let it receive a second impulsion capable of making it describe the line AB in the same time *t*; the material point will not entirely yield to the action of this second force, which tends to draw it in the direction of the line AB; since, by the law of inertia it would have described the line AC$=m$A in the time *t*, if the second impulsion had not been communicated to it; but it will describe the diagonal AD of the parallelogram ABDC. If it should receive at D a third impulse capable of moving it over the line DG in a third time *t*, it will, for a similar reason, describe the diagonal DF of a parallelogram constructed upon DG, and DE the prolongation of AD, &c.; thus, at the end of a time equal to *nt*, the material point will have described a polygon having *n* sides.

The velocity being constant so long as the material point remains on the same side of the polygon, it follows, that if at its arrival at the extremity of either side, it be not subjected to a new impulse, it will continue to move in the direction of this side, with a constant velocity.

428. If the time *t* be supposed indefinitely small, the impulsions will be communicated in consecutive instants, and the polygon will then be transformed into a curve.

The time *t* being supposed indefinitely small, it may be represented by dt, and the side of the polygon which is passed over in this time, will become the element of the curve: consequently, to determine the velocity, which will be measured by the space which the body would pass over in the direction of the tangent, in a unit of time, if the incessant force should cease to communicate new impulses, we must multiply ds, the element of the curve, by the number of times that dt is contained in unity; that is, we multiply ds by $\frac{1}{dt}$, and we thus obtain

$$v=\frac{ds}{dt}.$$

429. Let the body be supposed to describe the polygon m, m', m'', m''', &c. (*Fig.* 167), receiving increments to its velocity at the points m, m', m'', m''', &c. Let v, v', v'', v''', &c. represent the velocities which the body has acquired at the points m, m', m'', m''', &c., and t, t', t'', t''', &c. the times employed in describing the sides mm', $m'm''$, $m''m'''$, &c. Since each of these sides is supposed to be described with a constant velocity, we shall have, by the principles of uniform motion,

$$mm'=vt, \quad m'm''=v't', \quad m''m'''=v''t'', \text{ \&c.};$$

and the perimeter of the polygon will therefore be expressed by

$$vt+v't'+v''t''+\text{\&c.}$$

If we project the sides of this polygon on the co-ordinate axes, denoting by α, β, γ, α', β', γ', &c. the angles formed by the sides mm', $m'm''$, $m''m'''$, &c. with these axes, the projections of the sides will be expressed by

$vt\cos\alpha$, $v't'\cos\alpha'$, $v''t''\cos\alpha''$, &c., on the axis of x,

$vt\cos\beta$, $v't'\cos\beta'$, $v''t''\cos\beta''$, &c., on the axis of y,

$vt\cos\gamma$, $v't'\cos\gamma'$, $v''t''\cos\gamma''$, &c., on the axis of z;

and the projection $nn'n''n'''$, &c. of the perimeter $mm'm''m'''$, on the axis of x, will be expressed by

$$vt\cos\alpha+v't'\cos\alpha'+v''t''\cos\alpha''+\text{\&c.} \ldots\ldots (178).$$

It thus appears that while the material point m describes the polygon $mm'm''m'''$, &c., its projection n will describe the space $nn'n''n'''$, &c. But if the point n were merely solicited by a force X directed along the axis of x, and of such intensity that the point should describe the spaces nn', $n'n''$, $n''n'''$, &c., in the times t, t', t'', &c., with the velocities $v\cos\alpha$, $v'\cos\alpha'$, $v''\cos\alpha''$, &c., the space passed over on the axis of x would be expressed by

$$v\cos\alpha t+v'\cos\alpha' t'+v''\cos\alpha'' t''+\&c. \ldots\ldots (179).$$

In obtaining the expression (179), no reference has been had to the components of the velocity parallel to the axes of y and z; and the identity of the expressions (178) and (179) therefore proves that when the point m is transported in space, its projection moves on the axis of x, as though the other two components of the velocity did not exist.

The same remarks being applicable to the other two axes, and the polygon becoming a curve when the number of its sides is increased indefinitely, it follows that when a material point solicited by an incessant force describes a curve in space, each projection of the point moves independently of the motions of the other two.

Thus, by calling X, Y, and Z the components of the incessant force φ, parallel to the three axes, we can regard these components as forces which impress on the projections of the material point motions which are entirely independent of each other.

430. To determine the analytical expressions for these incessant forces, we remark, that while the material point describes the space ds, its projections describe the spaces dx, dy, and dz respectively: the velocities of the projections will therefore be represented by $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$; and since the incessant force is equal to the differential coefficient of the velocity considered as a function of the time, we shall have, by regarding dt as constant,

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\mathrm{X}\\ \frac{d^2y}{dt^2}&=\mathrm{Y}\\ \frac{d^2z}{dt^2}&=\mathrm{Z}\end{aligned}\right\} \ldots\ldots (180)$$

Such are the equations which serve to determine the circumstances of the motion of a material point describing a curve.

431. When the functions X, Y, and Z are given by the nature of the problem, and if the integrals of the equations (180) can be obtained, these integrals will give three relations between the four variables x, y, z, and t: the quantity t being eliminated, there will remain two relations between x, y, and z, which will represent the equations of the *trajectory*, *or curve described by the material point under the influence of the incessant forces.*

When the forces are situated in a single plane, which may be taken as that of x, y, the trajectory will be contained in the same plane, and it will then only be necessary to use the two equations

$$\frac{d^2x}{dt^2}=\text{X}, \quad \frac{d^2y}{dt^2}=\text{Y}.$$

When, by the nature of the problem, the quantities X and Y are known, and if the integrals of these equations can be obtained, they will contain no other variables than x, y, and t; thus, by eliminating t, we shall find a relation between x and y, which may be written under the following form,

$$y=fx;$$

this relation will be the equation of the plane curve described by the material point.

432. The velocity of the material point at any instant is expressed by

$$v=\frac{ds}{dt}:$$

but the element ds of the arc of a curve situated in space, being considered as an indefinitely small right line, whose projections on the co-ordinate axes are represented by dx, dy, and dz, the value of this element will be

$$\sqrt{(dx^2+dy^2+dz^2)}.$$

Substituting this value in the preceding equation, we have

$$v=\frac{1}{dt}\sqrt{(dx^2+dy^2+dz^2)},$$

or, since the differentials are taken with reference to t as a variable,

$$v=\sqrt{\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right)} \;\ldots\ldots (181).$$

The angles formed by the direction of the motion with the co-ordinate axes will result from the equations

$$v\cos\alpha=\frac{dx}{dt},$$

$$v\cos\beta=\frac{dy}{dt},$$

$$v\cos\gamma=\frac{dz}{dt}.$$

433. The velocity may likewise be determined in the following manner. Let the equations (180) be multiplied respectively by $2dx$, $2dy$, and $2dz$; the sum of these products will give

$$\frac{2dx\,.\,d^2x+2dy\,.\,d^2y+2dz\,.\,d^2z}{dt^2}=2(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz):$$

and since the first member is the differential of $dx^2+dy^2+dz^2$, divided by dt^2, we shall have

$$\frac{d(dx^2+dy^2+dz^2)}{dt^2}=2(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz);$$

or, replacing $dx^2+dy^2+dz^2$ by its value ds^2, and integrating, we obtain

$$\frac{ds^2}{dt^2}=2\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)+\mathrm{C};$$

and by substituting v for $\frac{ds}{dt}$, we find

$$v^2=2\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)+\mathrm{C} \;\ldots\ldots (182).$$

434. It thus appears that the determination of the velocity will depend on the integration of the expression

$$\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz) \;\ldots\ldots (183).$$

When this integration is possible, the integral will be a function of the variables x, y, and z, and the equation (182) may be written under the form

$$v^2=2\mathrm{F}(x,y,z)+\mathrm{C} \;\ldots\ldots (184).$$

To determine the value of the constant, we must know the velocity of the moveable point, at a given point of the trajectory. Thus, if V be the velocity at that point which corresponds to the co-ordinates $x=a$, $y=b$, $z=c$, we shall have

$$V^2=2F(a, b, c)+C.$$

The value of C being deduced from this equation, and substituted in equation (184), we shall obtain

$$v^2-V^2=2F(x, y, z)-2F(a, b, c).$$

435. The expression (183) is integrable when the moveable point is subjected to the action of a force which is constantly directed towards a fixed centre. To demonstrate this proposition, we will represent the resultant R of the several forces acting on the material point by CD, a portion of the line CM drawn from the point to the fixed centre (*Fig.* 168); let this centre be assumed as the origin of co-ordinates, and denote by λ the distance of the point M from the origin, and by α, β, γ the angles formed by CM with the axes of co-ordinates: the direction of the resultant forming the same angles, we shall have

$$X=R\cos\alpha,\quad Y=R\cos\beta,\quad Z=R\cos\gamma,$$

and consequently

$$\frac{X}{Y}=\frac{\cos\alpha}{\cos\beta},\quad \frac{Y}{Z}=\frac{\cos\beta}{\cos\gamma},\quad \frac{X}{Z}=\frac{\cos\gamma}{\cos\alpha}\ \ldots\ldots (185).$$

But if x, y, and z denote the co-ordinates of the point M, we shall have

$$x=\lambda\cos\alpha,\quad y=\lambda\cos\beta,\quad z=\lambda\cos\gamma;$$

whence, by division,

$$\frac{x}{y}=\frac{\cos\alpha}{\cos\beta},\quad \frac{y}{z}=\frac{\cos\beta}{\cos\gamma},\quad \frac{z}{x}=\frac{\cos\gamma}{\cos\alpha};$$

these values substituted in equations (185), give

$$yX-xY=0,\quad zY-yZ=0,\quad xZ-zX=0.$$

If in these equations we replace X, Y, and Z by their values deduced from equations (180), we shall find

$$y\frac{d^2x}{dt^2}-x\frac{d^2y}{dt^2}=0,$$

$$z\frac{d^2y}{dt^2}-y\frac{d^2z}{dt^2}=0,$$

$$x\frac{d^2z}{dt^2}-z\frac{d^2x}{dt^2}=0.$$

Multiplying the first of these equations by dt, integrating and reducing, we obtain

$$\frac{ydx-xdy}{dt}=C\,.\,.\,.\,.\,.\,(186).$$

The other two equations being treated in a similar manner, we find

$$ydx-xdy=Cdt,$$
$$zdy-ydz=C'dt,$$
$$xdz-zdx=C''dt.$$

If we multiply each of these equations by the variable which it does not contain, and take the sum of the products, there will result

$$dt(Cz+C'x+C''y)=0,$$

or,

$$Cz+C'x+C''y=0.$$

This equation being that of a plane passing through the origin of co-ordinates, or centre of attraction, it follows that the point will describe a plane curve.

In the resolution of this problem it will therefore be unnecessary to employ the equation $Z=\frac{d^2z}{dt^2}$, and it will simply be necessary to integrate the equation (186), which may be written thus:

$$ydx-xdy=Cdt;$$

and from this we deduce

$$\int(ydx-xdy)=Ct+C'\,.\,.\,.\,.\,.\,(187).$$

To determine the value of this integral, we remark that ydx being the element of a surface bounded by a curve, we can suppose this surface to be included within the limits $x=0$ and $x=CP$ (*Fig.* 169); thus, the expression $\int ydx$ will be represented by the area LCPM. If from this area we subtract the triangle CPM, there will remain

sector LCM=area LCPM−triangle CPM,

or,

$$\text{sector LCM}=\int ydx-\frac{xy}{2};$$

differentiating and reducing, we find

$$d(\text{sector LCM})=\frac{ydx-xdy}{2};$$

and again integrating,

$$2\,.\,\text{sector LCM}=\int(ydx-xdy):$$

hence, the equation (187) can be reduced to the following:

$$2\,.\,\text{sector LCM}=Ct \;.\;.\;.\;.\;.\; (188);$$

the constant C′ is here suppressed, since we may always regard the times as reckoned from the instant when the moveable point is situated at the point L, in which case the sector will become equal to zero.

If we make C=2A, the equation (188) will become

$$\text{sector LCM}=At;$$

from which we conclude, *that when a material point solicited by a force which is constantly directed towards a fixed centre, describes a curve* LM *about this centre, the area of the sector* LCM *described by the radius vector drawn to the material point is constantly proportional to the time which the point employs in describing the curve.* This property is called *the principle of areas proportional to the times.*

436. The formula (183) is always integrable when the forces are directed towards fixed centres, their intensities being at the same time functions of the distances of the material point from these centres.

Let M represent the place of the material point (*Fig.* 170), which is attracted by the forces P, P′, P″, &c. towards the fixed centres C, C′, C″, &c.: denote by

x, y, z, the co-ordinates of the point M,
a, b, c, the co-ordinates of the centre C,
a', b', c', the co-ordinates of the centre C′,
a'', b'', c'', the co-ordinates of the centre C″,
&c. &c. &c.
p, p', p'', &c., the distances CM, C′M, C″M, &c.;
α, β, γ, the angles formed by p with the axes of co-ordinates,
α', β' γ', the angles formed by p' with the same axes,
α'', β'', γ'', the angles formed by p'' with the same axes,
&c. &c. &c. &c.

The total resultant of the attractive forces will have the following components parallel to the three axes,

$$\left.\begin{array}{l}X=P\cos\alpha+P'\cos\alpha'+P''\cos\alpha''+\&c.\\ Y=P\cos\beta+P'\cos\beta'+P''\cos\beta''+\&c.\\ Z=P\cos\gamma+P'\cos\gamma'+P''\cos\gamma''+\&c.\end{array}\right\}\ldots\ldots(189).$$

The projection of the right line CM on the axis of x being represented by BD (*Fig.* 170), we have

$$BD=AB-AD;$$

and by observing that AB and AD are the co-ordinates x and a of the points M and C, and that BD, being the projection of MC on the axis of x, is expressed by $p\cos\alpha$, we shall find, by substituting these values in the preceding equation,

$$p\cos\alpha=x-a;$$

the same remarks being applicable to the projections on the other two axes, we shall have

$$p\cos\alpha=x-a,\quad p\cos\beta=y-b,\quad p\cos\gamma=z-c$$

And in like manner,

$$\begin{array}{lll}p'\cos\alpha'=x-a', & p'\cos\beta'=y-b', & p'\cos\gamma'=z-c',\\ p''\cos\alpha''=x-a'', & p''\cos\beta''=y-b'', & p''\cos\gamma''=z-c'',\\ \&c. & \&c. & \&c.\end{array}$$

By eliminating the cosines of these angles, the equations (189) become

$$X=P\frac{x-a}{p}+P'\frac{x-a'}{p'}+P''\frac{x-a''}{p''}+\&c.,$$

$$Y=P\frac{y-b}{p}+P'\frac{y-b'}{p'}+P''\frac{y-b''}{p''}+\&c.,$$

$$Z=P\frac{z-c}{p}+P'\frac{z-c'}{p'}+P''\frac{z-c''}{p''}+\&c.$$

These values substituted in the formula (183) give

$$\begin{aligned}\int(Xdx+Ydy+Zdz)=&\int P\left(\frac{x-a}{p}dx+\frac{y-b}{p}dy+\frac{z-c}{p}dz\right)\\ &+\int P'\left(\frac{x-a'}{p'}dx+\frac{y-b'}{p'}dy+\frac{z-c'}{p'}dz\right)\\ &+\&c.\qquad \&c.\qquad \&c.\ldots\ldots(190).\end{aligned}$$

But the distances of the point M from the centres C, C′, C″, &c. being given by the equations

$$\begin{array}{l}(x-a)^2+(y-b)^2+(z-c)^2=p^2,\\ (x-a')^2+(y-b')^2+(z-c')^2=p'^2,\\ \&c.\qquad\&c.\qquad\&c.,\end{array}$$

we shall obtain, by differentiating,

$$\frac{x-a}{p}dx+\frac{y-b}{p}dy+\frac{z-c}{p}dz=dp,$$

$$\frac{x-a'}{p'}dx+\frac{y-b'}{p'}dy+\frac{z-c'}{p'}dz=dp',$$

$$\&c. \qquad \&c. \qquad \&c.;$$

and substituting these values in equation (190), we find

$$\int(\text{X}dx+\text{Y}dy+\text{Z}dz)=\int(\text{P}dp+\text{P}'dp'+\text{P}''dp''+\&c.)\ldots.(191).$$

But the forces P, P′, P″, &c. are, by hypothesis, functions of the distances p, p', p'', &c.; the expression $\text{P}dp+\text{P}'dp'+\text{P}''dp''$ will therefore contain but a single variable in each term, and its integral may be effected by the method of quadratures.

It should be observed that the factors dp, dp', dp'', &c. may become negative, if the expressions $x-a$, $y-b$, $z-c$, $x-a'$, &c. should be transformed into $a-x$, $b-y$, $c-z$, $a'-x$, &c.

437. For the purpose of making an application of the preceding theorem, let it be required to determine the velocity of a material point which moves from rest, under the influence of a force of attraction which is constantly directed towards a fixed centre, and which varies in intensity in the inverse ratio of the square of the distance from the position of the point to the fixed centre. Let the direction of the force be supposed to coincide with the axis of z: the co-ordinate axes being disposed as in *Fig.* 171, the intensity of the force and the co-ordinate z will increase together, and we shall have

$$p=\text{AC}-\text{AM}=c-z, \quad dp=-dz.$$

If g represent the intensity of the force at the distance r from the centre C, and P its intensity at the distance p, we shall have the proportion

$$g:\text{P}::\frac{1}{r^2}:\frac{1}{p^2};$$

whence,

$$\text{P}=g\frac{r^2}{p^2};$$

but dp being negative, the quantity Pdp should be replaced by $-\frac{gr^2}{p^2}dp$; integrating, we reduce the equation (191) to

$$\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)=\frac{gr^2}{p}+\mathrm{C}.$$

This value being substituted in formula (182) gives

$$v^2=\frac{2gr^2}{p}+\mathrm{C}\ \ldots\ldots (192).$$

To determine the value of the constant C, we suppose the body to commence its motion at a point whose distance from the centre of attraction is represented by a; the velocity at this point being equal to zero, we have

$$0=\frac{2gr^2}{a}+\mathrm{C},$$

or,

$$\mathrm{C}=-\frac{2gr^2}{a};$$

the equation (192) will therefore become

$$v^2=2gr^2\left(\frac{1}{p}-\frac{1}{a}\right).$$

If a be regarded as the unit of distance, the value of v^2 will become identical with that determined in Art. 409.

438. To apply the formulas (180) we will first investigate the trajectory described by a material point which moves under the influence of a single impulse. In this case, the incessant forces being equal to zero, we shall have

$$\mathrm{X}=0,\quad \mathrm{Y}=0,\quad \mathrm{Z}=0;$$

and the equations (180) reduce to

$$\frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=0,\quad \frac{d^2z}{dt^2}=0;$$

multiplying by dt, they become

$$\frac{d^2x}{dt}=0,\quad \frac{d^2y}{dt}=0,\quad \frac{d^2z}{dt}=0.$$

The integrals of these equations are

$$\frac{dx}{dt}=a,\quad \frac{dy}{dt}=b,\quad \frac{dz}{dt}=c\ \ldots\ldots (193).$$

Substituting these values in equation (181), we find

$$v=\sqrt{(a^2+b^2+c^2)}=\text{a constant};$$

and denoting this constant by A, we have

$$\frac{ds}{dt}=A;$$

consequently,

$$s=At+B;$$

and the motion of the material point will be uniform.

The motion is likewise rectilinear; for the equations (193) give, by integration,

$$x=at+a', \quad y=bt+b', \quad z=ct+c',$$

whence, by eliminating t,

$$x=\frac{az}{c}+\frac{a'c-ac'}{c}, \quad y=\frac{bz}{c}+\frac{b'c-bc'}{c}.$$

These equations evidently appertain to the projections of a right line on the planes of x, z and y, z.

Of the Motion of a Material Point when compelled to describe a particular Curve.

439. When a material point m, without weight, has received an impulse K (*Fig.* 172), and is subjected to the condition of moving upon a particular curve, we can resolve this impulse into two components, one mN=K′ normal to the curve, the other mT=K″ in the direction of the tangent: the normal force will be destroyed by the resistance of the curve, and the tangential component will produce its entire effect in communicating motion to the material point.

If we regard the curve as a polygon $mm'm''m'''$, &c. (*Fig.* 173), having an infinite number of sides, the angle $tm'm''$ formed by the prolongation of the side mm' with the consecutive side $m'm''$ is called the angle of contact; it will be denoted by ω; the plane $tm'm''$ is the osculatory plane at the point m', and in plane curves coincides with the plane of the curve.

The material point m, being solicited by a force K, receives a primitive velocity v, causing it to describe the side mm'; but having arrived at the point m', it is deflected from its course, and describes the side $m'm''$. By this deflection it necessarily undergoes a loss of velocity which will now be estimated.

For this purpose, let the velocity v be represented by the line $m'q$. This velocity being resolved into two components $m'n$ and $m'l$, respectively parallel and perpendicular to the side $m'm''$, we shall have

$$m'l=m'q\,.\sin tm'm'',\quad m'n=m'q\,.\cos tm'm'',$$

or,

$$m'l=v\,.\sin\omega,\quad m'n=v\,.\cos\omega.$$

The component $v\,.\sin\omega$ being destroyed by the resistance of the polygon, the velocity v will be reduced to $v\,.\cos\omega$; and consequently, the velocity lost, being equal to the primitive velocity diminished by the velocity actually remaining, will be expressed by $v(1-\cos\omega)$.

When the polygon is supposed to become a curve, the angle $tm'm''$ becomes infinitely small, and the quantity $v(1-\cos\omega)$ is at the same time an infinitely small quantity of the second order.

To prove that this is the case, we observe that $1-\cos\omega$ represents the versed sine DB of an angle ω (*Fig.* 174), measured by the arc BC; and we have the proportion

$$\mathrm{AD}:\mathrm{CD}::\mathrm{CD}:\mathrm{DB}.$$

But when the arc CB becomes infinitely small, CD will be so likewise; and since CD is then infinitely small with respect to AD, it follows from the above proportion, that DB must be infinitely small with respect to CD, or that it is an infinitely small quantity of the second order. Thus, the velocity lost at each side of the polygon being an infinitely small quantity of the second order, it may be neglected, since the sum of these velocities, although infinite in number, will constitute but an infinitely small quantity of the first order, which may be neglected in comparison with the original velocity v. Hence, we conclude, that a material point which is compelled to describe a curve, preserves undiminished the velocity which was originally communicated to it.

440. The component of the velocity $v\,.\sin\omega$ with which the material point is pressed against the curve, and which is destroyed by the curve's resistance, varies constantly as the

point changes its position, since $\sin \omega$ is constantly variable: we may regard this resistance exerted by the curve as an incessant force constantly acting upon the point and deflecting it from the tangent along which it would otherwise tend to move.

When there are several forces acting on the material point, we resolve each in a similar manner, and the sum of the normal components must then be added to the pressure arising from the component of the velocity.

441. Let it be supposed that a force N equal and directly opposed to the resultant of all the normal forces is applied to the material point: this force will produce precisely the same effect as the resistance offered by the curve, and the latter will therefore be represented by N. Let α, β, γ be the angles formed by the direction of the force N with the co-ordinate axes; the components of N in the direction of the axes will be respectively

$$N\cos\alpha, \quad N\cos\beta, \quad N\cos\gamma,$$

and should be added to the components of the incessant forces in the general equations of motion (180): we shall thus obtain

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X + N\cos\alpha \\ \frac{d^2y}{dt^2} &= Y + N\cos\beta \\ \frac{d^2z}{dt^2} &= Z + N\cos\gamma \end{aligned}\right\} \dots\dots (194).$$

To these equations may be added two others which result from the relations existing between the angles α, β, and γ; the first of these equations is

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \dots\dots (195).$$

The second is

$$\cos\alpha . \cos\alpha' + \cos\beta . \cos\beta' + \cos\gamma . \cos\gamma' = 0,$$

in which α', β', γ' represent the angles formed by the tangent to the curve with the co-ordinate axes. The cosines of these last angles may be expressed as follows:

$$\cos\alpha' = \frac{dx}{ds}, \quad \cos\beta' = \frac{dy}{ds}, \quad \cos\gamma' = \frac{dz}{ds};$$

these values substituted in the preceding equation convert it into

$$\frac{dx}{ds}\cos\alpha+\frac{dy}{ds}\cos\beta+\frac{dz}{ds}\cos\gamma=0\ \ldots\ldots\ (196).$$

442. To determine the velocity of the material point, let the equations (194) be multiplied respectively by $2dx$, $2dy$, and $2dz$: their sum being taken, we shall obtain

$$2dx\frac{d^2x}{dt^2}+2dy\frac{d^2y}{dt^2}+2dz\frac{d^2z}{dt^2}=2(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)$$
$$+2\mathrm{N}(dx\,.\cos\alpha+dy\,.\cos\beta+dz\,.\cos\gamma):$$

the last term of this equation being equal to zero, by formula (196), there remains

$$2dx\frac{d^2x}{dt^2}+2dy\frac{d^2y}{dt^2}+2dz\frac{d^2z}{dt^2}=2(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz);$$

or,

$$\frac{d(dx^2+dy^2+dz^2)}{dt^2}=2(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz):$$

whence, by substitution and integration, we find

$$\frac{ds^2}{dt^2}=2\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)+\mathrm{C};$$

or,

$$v^2=2\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)+\mathrm{C}\ \ldots\ldots\ (197).$$

443. When the material point merely receives an impulse, without being acted upon by incessant forces, we have

$$\mathrm{X}=0,\quad \mathrm{Y}=0,\quad \mathrm{Z}=0;$$

and consequently,

$$v^2=\text{a constant.}$$

Thus, when the material point is compelled to describe a curve, being acted upon only by an impulse, its velocity will remain invariable. This result accords with that which has been already obtained (Art. 438), on the supposition that the motion is perfectly free.

444. Let the material point which is supposed to describe the curve, be acted on by the force of gravity; we shall then have

$$\mathrm{X}=0,\quad \mathrm{Y}=0,\quad \mathrm{Z}=g;$$

and the equation (197) will be reduced to

$$v^2=2\int gdz+\mathrm{C}.$$

If the velocity v be supposed equal to V, when $z=0$, we shall find

$$V^2=C.$$

This value substituted in the preceding equation gives

$$v^2=2gz+V^2\ ;$$

whence,

$$v=\sqrt{(2gz+V^2)} \ldots\ldots (198).$$

This expression for the velocity being independent of the relations which may exist between the co-ordinates x, y, and z, it follows that the velocity will be the same for the same value of z, whatever may be the form of the curve.

To determine the expression for the time employed by the material point in describing a given portion of the curve, we replace v by its value $\frac{ds}{dt}$, and thus obtain

$$\frac{ds}{dt}=\sqrt{(2gz+V^2)}\ ;$$

whence,

$$dt=\frac{ds}{\sqrt{(2gz+V^2)}} \ldots\ldots (199)\ ;$$

or, by substitution,

$$dt=\frac{\sqrt{(dx^2+dy^2+dz^2)}}{\sqrt{(2gz+V^2)}} \ldots\ldots (200).$$

To integrate this equation, it will be necessary to reduce it, by means of the equations of the curve, to one which shall contain but two variables; thus, if the equations of the curve are

$$f(x, z)=0, \quad f(y, z)=0 \ldots\ldots (201),$$

we may, by the aid of these equations, in connexion with equation (200), eliminate two of the three variables x, y, and z; and it will then only be necessary to integrate an equation expressing the relation between dt and one of the co-ordinates of the moveable point.

445. If, for example, the curve be supposed to become a right line, the equations (201) will be of the form

$$x=az+\alpha, \quad y=bz+\beta \ldots\ldots (202):$$

from which we deduce

$$dx=adz, \quad dy=bdz\ ;$$

and by substituting these values in the formula (200), it is transformed into

$$dt=\frac{dz\sqrt{(1+a^2+b^2)}}{\sqrt{(2gz+V^2)}}.$$

If the point be supposed to move from rest, its initial velocity V will be equal to zero, and we shall have, by division,

$$\frac{dt}{\sqrt{(1+a^2+b^2)}}=\frac{dz}{\sqrt{(2gz)}};$$

whence, by integration,

$$\frac{t}{\sqrt{(1+a^2+b^2)}}=\frac{1}{g}\sqrt{(2gz)}\ \ldots\ldots\ (203).$$

The constant introduced by integration becomes equal to zero, since, by hypothesis, when $t=0$, $v=0$, and $z=0$ (Art. 444).

446. To determine the space passed over in the time t, we suppress V in equation (199), which then becomes

$$\frac{ds}{dt}=\sqrt{(2gz)},$$

and eliminating z by means of equation (203), there results

$$ds=\frac{gt\,.\,dt}{\sqrt{(1+a^2+b^2)}};$$

and by integration,

$$s=\frac{\frac{1}{2}gt^2}{\sqrt{(1+a^2+b^2)}}+C:$$

which proves that the motion is similar to that of a body on an inclined plane, as might have been anticipated.

447. The co-ordinates x, y, and z are readily determined in functions of the time; for the latter is given by formula (203), and this, taken in connexion with equations (202), will determine x and y in functions of t.

448. If, as in the present instance, the point be supposed to describe a plane curve, and if the incessant forces act entirely in this plane, we may, by placing the axes of x and y in this plane, dispense with the consideration of the third of equations (194); the formulas (195) and (196) will then be reduced to

$$\cos^2\alpha+\cos^2\beta=1,\quad \frac{dx}{ds}\cos\alpha+\frac{dy}{ds}\cos\beta=0;$$

and the two equations of the curve will be replaced by the single relation

$$y=fx.$$

449. The velocity being given by formula (198), without the aid of equations (201), we conclude that the velocity acquired by the moveable point is independent of the form of the curve, being determined by the value of the vertical ordinate. Consequently, if from the point O (*Fig.* 175), where $z=0$, and $v=V$, we draw the arcs of different curves OM, OM′, OM″, &c., terminated by the horizontal plane KL, the ordinates z of the first and last points of all these arcs being equal, it follows that different bodies departing from the point O with the common velocity V, and describing the several curves, will all have acquired the same velocity when they shall have arrived at the points M, M,′ M″, &c., situated in the same horizontal plane.

450. In general, whatever may be the number of forces acting on the moveable point, if the equation (197) be integrable, we can determine the velocity v without knowing the nature of the curve described. For, the values of the incessant forces X, Y, and Z, expressed in functions of the co-ordinates x, y, and z, being substituted in equation (197), if the expression

$$\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)$$

becomes integrable, we may represent it by

$$f(x, y, z);$$

and the equation (197) will then reduce to

$$v^2=2f(x, y, z)+\mathrm{C}.$$

If we denote by a, b, and c the values of x, y, and z which correspond to the velocity V, the value of C will become known; thus,

$$\mathrm{C}=\mathrm{V}^2-2f(a, b, c);$$

and replacing C by this value in the general expression for the velocity, we find

$$v^2=\mathrm{V}^2+2f(x, y, z)-2f(a, b, c);$$

an expression which depends only on the initial velocity, and the co-ordinates of the first and last points of the curve described.

451. It has been explained (Art. 440) that the normal pressure exerted against the curve arises in part from the normal components of the incessant forces, and partly from the normal force due to the velocity. To determine the value of the latter, let perpendiculars on and on' be erected at the middle points of the equal consecutive sides mm' and $m'm''$ (*Fig.* 176) of the polygon having an infinite number of sides: the angle $tm'm''$, formed by one of these sides with the prolongation of the other, will be the angle which we have represented by ω. But the angles n and n' being right angles, we have

$$non'+nm'n'=180^\circ=tm'm''+nm'n';$$

and therefore,

$$tm'm''=\omega=non'=2nom'.$$

The angle nom' being infinitely small, its sine may be regarded as equal to the arc which measures it; but this sine is expressed by $\frac{m'n'}{m'o}$, or $\frac{m'n'}{no}$, since no and $m'o$ may be considered equal; hence,

$$\omega=\frac{2m'n}{no}=\frac{mm'}{no}.$$

If we now return to the consideration of the curve which is the limit of the polygon, the side mm' becomes the element of the curve, and no the radius of curvature: the preceding relation will therefore be transformed into

$$\omega=\frac{ds}{\gamma},$$

γ denoting the radius of curvature.

Let φ denote the intensity of the incessant force which is due to the normal component of the velocity: this intensity being in general expressed by the quotient of the element of the velocity, divided by the element of the time, and the element of the velocity being represented in the present instance by $v.\sin\omega$, we shall have

$$\varphi=\frac{v.\sin\omega}{dt},$$

or, since the infinitely small arc may be substituted for its sine, this expression becomes

$$\varphi=\frac{v\omega}{dt};$$

replacing ω by its value found above, we have

$$\varphi=\frac{vds}{\gamma dt}, \text{ or } \varphi=\frac{v^2}{\gamma}.$$

The normal pressure resulting from the other forces may be determined by the parallelogram of forces, and this pressure must then be combined with that due to the velocity, in order to obtain the total pressure.

452. Let it be supposed, for example, that the material point describes a plane curve, and that the incessant forces are directed in the plane of this curve: let these forces be reduced to their resultant R, and denote by θ the angle formed by the direction of the resultant with that part of the normal which lies on the concave side of the curve: the component of the resultant in the direction of the normal will be expressed by R cos θ, and will act in the same or in a contrary direction to the pressure arising from the velocity, according as θ is obtuse or acute. The pressure arising from the velocity being always directed *from* the centre of curvature, the entire pressure will be expressed by

$$N=\frac{v^2}{\gamma}-R\cos\theta;$$

this pressure will be exerted *from* the centre of curvature so long as the quantity N is positive, and *towards* the centre in the contrary case.

Of the Motion of a material Point when compelled to move upon a Curved Surface.

453. When a material point which is compelled to move upon a curved surface is subjected to the action of incessant forces, these forces, and that resulting from the velocity of the point, will exert a pressure against the surface, which will be counteracted by the resistance of the surface. If we denote this resistance by N, the material point may be regarded as moving freely in space, provided we include the components of the force N in the general equations (180), which express the circumstances of motion of a point under

the influence of incessant forces. Let α, β, and γ represent the angles formed by the direction of the force N with the co-ordinate axes; its components in the directions of these axes will be expressed by N cos α, N cos β, and N cos γ: consequently, the equations of the required motion will be

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\mathrm{X}+\mathrm{N}\cos\alpha\\ \frac{d^2y}{dt^2}&=\mathrm{Y}+\mathrm{N}\cos\beta\\ \frac{d^2z}{dt^2}&=\mathrm{Z}+\mathrm{N}\cos\gamma\end{aligned}\right\}\ldots\ldots(204).$$

The angles α, β, and γ will become known when the equation of the surface $\mathrm{L}=0$ is given, for we have, (Art. 62),

$$\cos\alpha=\pm\frac{\dfrac{d\mathrm{L}}{dx}}{\sqrt{\left(\dfrac{d\mathrm{L}}{dx}\right)^2+\left(\dfrac{d\mathrm{L}}{dy}\right)^2+\left(\dfrac{d\mathrm{L}}{dz}\right)^2}},$$

$$\cos\beta=\pm\frac{\dfrac{d\mathrm{L}}{dy}}{\sqrt{\left(\dfrac{d\mathrm{L}}{dx}\right)^2+\left(\dfrac{d\mathrm{L}}{dy}\right)^2+\left(\dfrac{d\mathrm{L}}{dz}\right)^2}},$$

$$\cos\gamma=\pm\frac{\dfrac{d\mathrm{L}}{dz}}{\sqrt{\left(\dfrac{d\mathrm{L}}{dx}\right)^2+\left(\dfrac{d\mathrm{L}}{dy}\right)^2+\left(\dfrac{d\mathrm{L}}{dz}\right)^2}};$$

the double signs prefixed to the values of the cosines, indicate that they may refer to the direction of a force which tends to pull the point, either along the normal to the surface, or along its prolongation.

If we put, for brevity,

$$\pm\frac{1}{\sqrt{\left(\dfrac{d\mathrm{L}}{dx}\right)^2+\left(\dfrac{d\mathrm{L}}{dy}\right)^2+\left(\dfrac{d\mathrm{L}}{dz}\right)^2}}=\mathrm{V},$$

the preceding equations will become

$$\cos\alpha=\mathrm{V}\frac{d\mathrm{L}}{dx},\quad \cos\beta=\mathrm{V}\frac{d\mathrm{L}}{dy},\quad \cos\gamma=\mathrm{V}\frac{d\mathrm{L}}{dz};$$

these values substituted in equations (204), reduce them to

$$\left.\begin{aligned}\frac{d^2x}{dt^2}&=\mathrm{X}+\mathrm{NV}\frac{d\mathrm{L}}{dx}\\ \frac{d^2y}{dt^2}&=\mathrm{Y}+\mathrm{NV}\frac{d\mathrm{L}}{dy}\\ \frac{d^2z}{dt^2}&=\mathrm{Z}+\mathrm{NV}\frac{d\mathrm{L}}{dz}\end{aligned}\right\}\ldots\ldots(205).$$

If N be eliminated between these equations, V will likewise disappear, and we shall thus obtain two relations, which, taken in connexion with the equation of the surface $\mathrm{L}=0$, will determine the co-ordinates of the moveable point in functions of the time.

454. As an example:—*Let it be required to determine the circumstances of the motion of a material point on the surface of a sphere:* let the origin of co-ordinates be assumed at the centre, the plane of x, y being horizontal, and the co-ordinates z being reckoned positive downwards; these co-ordinates will then be affected with the same sign as the force of gravity.

The equation of the sphere being

$$\mathrm{L}=x^2+y^2+z^2-a^2=0\ldots\ldots(206),$$

we obtain by differentiation,

$$d\mathrm{L}=2xdx+2ydy+2zdz=0\ldots\ldots(207),$$

and consequently,

$$\frac{d\mathrm{L}}{dx}=2x,\quad \frac{d\mathrm{L}}{dy}=2y,\quad \frac{d\mathrm{L}}{dz}=2z.$$

$$\mathrm{V}=\pm\frac{1}{\sqrt{(4x^2+4y^2+4z^2)}}=\pm\frac{1}{2a};$$

or,

$$\cos\alpha=\pm\frac{x}{a},\quad \cos\beta=\pm\frac{y}{a},\quad \cos\gamma=\pm\frac{z}{a}\ldots\ldots(207\,a).$$

Again, the force of gravity being the only incessant force acting on the material point, we have

$$\mathrm{X}=0,\quad \mathrm{Y}=0,\quad \mathrm{Z}=g;$$

these values reduce the equations (205) to

$$\frac{d^2x}{dt^2}=\pm N\frac{x}{a},\quad \frac{d^2y}{dt^2}=\pm N\frac{y}{a},\quad \frac{d^2z}{dt^2}=\pm N\frac{z}{a}+g\ .\ .\ .\ .\ .\ (208).$$

The positive signs should be taken together, and evidently correspond to like signs in the values of the cosines of α, β, and γ; a similar remark is applicable to the negative signs.

We eliminate $\pm$N between the two first of these equations, by multiplying them respectively by y and x, and taking their difference; we thus obtain, after multiplying by dt,

$$\frac{yd^2x-xd^2y}{dt}=0,\ \text{or}\ \frac{d(ydx-xdy)}{dt}=0;$$

whence, by integration, dt being regarded as constant,

$$ydx-xdy=Cdt\ .\ .\ .\ .\ .\ (209).$$

A second relation between the variables may be found by multiplying each of the equations (208) by the differential of the variable which it contains; the sum of these products will give

$$\frac{dx\,.\,d^2x+dy\,.\,d^2y+dz\,.\,d^2z}{dt^2}=\pm\frac{N}{a}(xdx+ydy+zdz)+gdz;$$

and since the quantity included within the brackets is equal to zero, by equation (207), the preceding result will be reduced to

$$\frac{dx\,.\,d^2x+dy\,.\,d^2y+dz\,.\,d^2z}{dt^2}=gdz:$$

multiplying by 2, and integrating, we have

$$\frac{dx^2+dy^2+dz^2}{dt^2}=2gz+C'\ .\ .\ .\ .\ .\ (210).$$

If two of the three variables x, y, and z be eliminated between the relations (206), (209), and (210), the result will be an equation which, being integrated, will give a relation between the third co-ordinate and the time t: this result will evidently be independent of the normal force, which has already disappeared from these three equations.

455. The equations (207) and (209) being squared, give

$$x^2dx^2+2xydxdy+y^2dy^2=z^2dz^2,$$
$$y^2dx^2-2xydxdy+x^2dy^2=C^2dt^2.$$

The sum of these equations being taken, the middle terms of the first members will disappear, and we shall have

$$(x^2+y^2)(dx^2+dy^2)=C^2dt^2+z^2dz^2;$$

substituting for (x^2+y^2) its value deduced from equation (206), there results

$$dx^2+dy^2=\frac{C^2dt^2+z^2dz^2}{a^2-z^2};$$

and eliminating dx^2+dy^2 between this equation and (210), we find

$$dt=\frac{adz}{\sqrt{[(a^2-z^2)(2gz+C)-C^2]}} \dots\dots (211).$$

The integral of this equation, which can only be obtained by approximation, will give the value of z in functions of the time.

456. To determine the expressions for the other co-ordinates in functions of the time, we will suppose ft to represent the approximate value of z determined from the integration of the preceding equation: if this value be substituted in equation (210), we may, by combining the resulting equation with that designated as (209), obtain two relations, the first between x and t, the second between y and t: but as the variables in each of these equations would not be separated by this process, we adopt another method of determining the values of x and y in functions of t.

Let AC$=x$, DC$=y$, mD$=z$ (*Fig.* 177) be the three co-ordinates of the point m on the surface of the sphere; if for a given value of z, the angle CAD, formed by the projection AD of the right line AM with the axis of x, were known, the corresponding values of x and y might be readily determined in functions of z. For, the angle CAD being denoted by θ, and the radius Am by a, we shall have AD$=\sqrt{(a^2-z^2)}$; and the triangle ACD right-angled at C, will give

$$\text{AC}=\text{AD}.\cos\text{CAD},\quad \text{CD}=\text{AD}.\sin\text{CAD};$$

or,

$$x=\sqrt{(a^2-z^2)}\times\cos\theta,\quad y=\sqrt{(a^2-z^2)}\times\sin\theta \dots\dots (212).$$

These two equations establishing a relation between x, y, and z, may be considered as replacing the equation of the sphere, which can be obtained by taking the sum of their squares. An additional variable θ is here introduced, but the number of relations is likewise increased by unity.

From the equations (212) we obtain by differentiation,

$$\left.\begin{array}{l} dx=-\sin\theta d\theta\sqrt{(a^2-z^2)}-\dfrac{zdz}{\sqrt{(a^2-z^2)}}\cos\theta \\ dy=\cos\theta d\theta\sqrt{(a^2-z^2)}-\dfrac{zdz}{\sqrt{(a^2-z^2)}}\sin\theta \end{array}\right\}\ldots\ldots(213):$$

multiplying the first of equations (213) by the second of (212), and the second of (213) by the first of (212), and taking their difference, we obtain

$$ydx-xdy=-(a^2-z^2)(\sin^2\theta+\cos^2\theta)d\theta;$$

or,

$$ydx-xdy=(z^2-a^2)d\theta.$$

This value substituted in equation (209), gives

$$(z^2-a^2)d\theta=\mathrm{C}dt,$$

and consequently,

$$d\theta=\frac{\mathrm{C}dt}{z^2-a^2};$$

or, replacing dt by its value deduced from equation (211), we obtain

$$d\theta=\frac{a\,.\,\mathrm{C}\,.\,dz}{(z^2-a^2)\sqrt{[(a^2-z^2)(2gz+\mathrm{C}')-\mathrm{C}^2]}}$$

This equation, being integrated by approximation, will determine the value of θ: we thence deduce the values of $\cos\theta$, and $\sin\theta$, which substituted in equations (212), determine the values of x and y in functions of z, and consequently in functions of the time t.

457. The equation (210) proves that the velocity is independent of the normal pressure; for, we deduce from that equation,

$$\frac{ds^2}{dt^2}=2gz+\mathrm{C}',$$

or,

$$v^2=2gz+\mathrm{C}';$$

and consequently,

$$v=\sqrt{(2gz+\mathrm{C}')}.$$

To determine the value of the normal pressure, we must recur to equations (208): these being multiplied respectively by x, y, and z, and added, give

$$\frac{xd^2x+yd^2y+zd^2z}{dt^2}=\pm\frac{\mathrm{N}}{a}(x^2+y^2+z^2)+gz \ldots\ldots (214).$$

But the differential of equation (207), $xdx+ydy+zdz=0$, being taken, we find

$$\frac{xd^2x+yd^2y+zd^2z}{dt^2}=-\frac{dx^2+dy^2+dz^2}{dt^2}=-v^2:$$

and this value substituted in equation (214) gives, after replacing $x^2+y^2+z^2$ by a^2,

$$-v^2=\pm\mathrm{N}a+gz;$$

or,

$$\pm\mathrm{N}=-\frac{v^2}{a}-\frac{gz}{a}.$$

458. If the moveable point be supposed situated at any instant below the horizontal plane passing through the centre of the sphere, the ordinate z will be positive, and the value of ±N becomes negative: and since N, which denotes the intensity of a force, is by hypothesis an essentially positive quantity, the inferior sign must be taken in order that —N may be essentially negative. Hence, it will be necessary to take the inferior signs in equations (208), and also in equations (207 a). The resistance of the surface will therefore be directed towards the centre, or the material point must be regarded as situated upon the *concave* surface of the sphere.

When the material point rises above the horizontal plane of x, y, the ordinate z will become negative, and the quantity $-v^2-gz$ *may* then become positive. In such case, the superior signs must be taken in equations (207 a) and (208), and the resistance of the surface will be exerted from the centre, or the body must be supposed to be on the *convex* surface.

The pressure exerted against the surface will be equal and directly opposed to the resistance which it offers, and will therefore be represented by $\frac{v^2+gz}{a}$ without reference to the sign of z.

If the moveable point be retained upon the surface of the sphere by an inflexible thread connecting the point with the centre, this thread will experience a *tension* so long as v^2+gz is positive; but, on the contrary, there will be a tendency to compress the thread whenever v^2+gz becomes negative.

Of the Motion of a material Point on the Arc of a Cycloid.

459. Let a material point M (*Fig.* 178) be supposed to move from rest on the arc of a cycloid, under the influence of the force of gravity: the initial velocity being by hypothesis equal to zero, the equation (198) is reduced to

$$v^2=2gz,$$

or

$$\frac{ds^2}{dt^2}=2gz;$$

whence we deduce

$$dt=\frac{ds}{\sqrt{(2gz)}}.$$

Let the origin of co-ordinates be assumed at the point E, the absciss ED of the point M′ being denoted by u, and the absciss EC of the point of departure by h: we shall then have

$$\text{CD}=\text{EC}-\text{ED};$$

or,

$$z=h-u.$$

This value being substituted in the preceding equation gives

$$dt=\frac{ds}{\sqrt{[2g(h-u)]}}\;.\;.\;.\;.\;.\;(215).$$

This equation contains three variables; we must therefore eliminate one by means of the equation of the cycloid. For this purpose, let $2a$ represent the diameter BE of the generating circle, and x and y the co-ordinates AP and PM′ of the point M′, reckoned from A as an origin; the equation of the curve will then be

$$dx=\frac{ydy}{\sqrt{(2ay-y^2)}}\;.\;.\;.\;.\;.\;(216).$$

But if s denote the arc AM′, we shall have the relation

$$ds=\sqrt{(dx^2+dy^2)};$$

or,

$$ds=dy\sqrt{\left(1+\frac{dx^2}{dy^2}\right)},$$

substituting in this equation the value of $\frac{dx}{dy}$ deduced from the relation (216), we find

$$ds=dy\sqrt{\left(1+\frac{y^2}{2ay-y^2}\right)};$$

or,

$$ds=dy\sqrt{\frac{2a}{2a-y}} \; \ldots\ldots \; (217).$$

But from an inspection of the figure, we have

$$2a-y=u;$$

and hence,

$$dy=-du.$$

By substituting these values in (217), we obtain

$$ds=-du\sqrt{\frac{2a}{u}}.$$

The differentials of s and u have contrary signs, since the first is a decreasing function of the second.

The preceding value of ds will reduce equation (215) to

$$dt=-\sqrt{\frac{a}{g}}\cdot\frac{du}{\sqrt{(hu-u^2)}} \; \ldots\ldots \; (218).$$

460. This equation may be integrated by the formula

$$\int\frac{dx}{\sqrt{(2x-x^2)}}=\text{arc (versed sine }=x);$$

for by making $x=\frac{z}{a}$, this formula becomes

$$\int\frac{dz}{\sqrt{(2az-z^2)}}=\text{arc}\left(\text{versed sine }=\frac{z}{a}\right) \; \ldots\ldots \; (219);$$

and consequently, by referring the integral of equation (218) to this formula, we obtain

$$t=-\sqrt{\frac{a}{g}}\,.\,\text{arc}\left(\text{versed sine}=\frac{u}{\frac{1}{2}h}\right)+\mathrm{C} \; \ldots\ldots \; (220).$$

To determine the constant, we remark, that since the time is reckoned from the instant when the body is at the point M, we must then have

$$t=0, \text{ and } u=\mathrm{EC}=h;$$

this supposition reduces the equation (220) to

$$0=-\sqrt{\frac{a}{g}}\,.\,\text{arc (versed sine }=2)+\text{C}.$$

But the arc whose versed sine is equal to 2, being a semi-circumference, if we denote by π the semi-circumference of a circle whose radius is unity, the preceding equation will become

$$\text{C}=\pi\sqrt{\frac{a}{g}}.$$

This value will reduce equation (220) to

$$t=\sqrt{\frac{a}{g}}\left(\pi-\text{arc (versed sine }=\frac{2u}{h}\right).$$

This expression gives the time of descent to the point M′, the absciss of which is equal to u. To obtain the entire time of descent to the vertex E, we make $u=0$, and the value of t is then reduced to

$$t=\pi\sqrt{\frac{a}{g}}$$

This value of the time being independent of the height h of the point of departure, we conclude *that the time necessary for a material point to descend to the vertex* E *of the cycloid, under the influence of the force of gravity, is constantly the same, whatever may be the position of its point of departure.*

Of Oscillatory Motion.

461. Let OBC (*Fig.* 179) represent a continuous curve, intersected at the points O and C by a horizontal line, and supposed to contain no angular points that might occasion a loss of velocity to a body or material point moving upon it. Let the tangent BT at the point B be supposed horizontal, the co-ordinate plane of x, y being likewise horizontal. If the co-ordinates z be reckoned positive downwards, we shall have the following equations to determine the circumstances of the motion of a material point sliding along the curve under the influence of gravity:

$$\frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=0,\quad \frac{d^2z}{dt^2}=g.$$

To determine the velocity of the moveable point, we proceed as in Art. 433: multiplying these equations by $2dx$, $2dy$, and $2dz$ respectively, and adding, we find

$$\frac{2dxd^2x+2dyd^2y+2dzd^2z}{dt^2}=2gdz;$$

and by integration,

$$\frac{dx^2+dy^2+dz^2}{dt^2}=2gz+C;$$

or,

$$\frac{ds^2}{dt^2}=2gz+C.$$

Replacing $\frac{ds^2}{dt^2}$ by its value v^2, there will result

$$v^2=2gz+C.$$

If V denote the velocity at the point O, when $z=0$, the preceding equation will become

$$C=V^2;$$

and consequently, by substituting this value of C, we shall obtain

$$v^2=V^2+2gz \;.\;.\;.\;.\;.\; (221).$$

462. Since the ordinates increase from the point O to the point B, it appears from equation (221) that the motion will be accelerated while the material point is describing the arc OB, and that its velocity will be a maximum at the point B: the ordinates decreasing beyond this point, the velocity of the moveable point will likewise be diminished. This diminution must be such that the material point will have at the point m', the same velocity as it previously had at the point m, situated in the same horizontal plane; for the vertical ordinates of these points being equal, their values substituted in equation (221) will necessarily give the same values for the two velocities.

The velocity diminishing constantly with the arc Om, we shall find on the prolongation of this arc a point A at which this velocity will have been equal to zero; and the moveable point may therefore be considered as moving from rest at this point. If through the point A a horizontal line AA′ be drawn, intersecting the second branch of the curve at A′, the velocity at A′ will likewise be equal to zero. Thus, the motion will cease at the point A′, and the action of gravity, causing it to descend from A′ to B, will augment the velocity in the same manner that it was before diminished. At the point B the velocity will again become a maximum, and the moveable point will then ascend to the point of departure A, its motion being retarded in the same manner that it was before accelerated in descending from A to B.

The same effects being repeated by the action of gravity, the point will continue to oscillate indefinitely.

If the arcs AB and A′B are similar, the times of describing them will evidently be equal. When the oscillations of a body or material point are all performed in equal times, they are said to be *isochronal.*

463. Let B′OBO′ (*Fig.* 180) represent a curve returning into itself, and symmetrical with respect to a vertical axis passing through the points B and B′ at which the tangents are horizontal. If the material point descend from a point O, with an initial velocity such that upon arriving at B it can ascend from B to B′ on the second branch of the curve, it will descend a second time on the arc B′OB, the force of gravity restoring the velocity lost during the ascent on the arc BO′B′. The same effects being repeated, the body will continue to revolve indefinitely.

Of the Simple Pendulum.

464. The simple pendulum is composed of a material heavy point M (*Fig.* 181), suspended by an inflexible right line MC devoid of weight, and oscillating about a point C. In this motion the point M describes the arc of a circle about C as a centre, and the velocity of M will be given (Art. 444) by the equation

$$v^2 = V^2 + 2gz \; \ldots\ldots \; (222).$$

Replacing v by its value $\frac{ds}{dt}$, we find

$$dt = \frac{ds}{\sqrt{(V^2 + 2gz)}} \; \ldots\ldots \; (223).$$

The origin being assumed at the point of departure, z will represent the ordinate M′P′ (*Fig.* 182) of the point M′, at which the material point is found after the lapse of a certain time, and V^2 will represent the square of the velocity which the body has at the point M, where $z=0$. If h denote the height due to this velocity, we shall have the relation

$$V^2=2gh,$$

and the equations (222) and (223) therefore become

$$v=\sqrt{[2g(h+z)]},\quad dt=\frac{ds}{\sqrt{[2g(h+z)]}}\,\ldots\ldots\,(224)$$

465. To express the quantity z in functions of the co-ordinates of the circle described with the radius CM, we demit the perpendiculars MB and M′D on the vertical line CE, and denote by a the radius CE, by b the vertical distance EB, and by x the absciss ED of the point M′ referred to the point E as an origin; we shall then have

$$z=\text{BD}=b-x.$$

And by introducing this value into equations (224), they become

$$v=\sqrt{[2g(h+b-x)]},\quad dt=\frac{ds}{\sqrt{[2g(h+b-x)]}}\,\ldots\ldots\,(225).$$

From the first of these relations we obtain the velocity of the material point at the point M′, corresponding to the absciss x; the second, being integrated, will determine the time employed by M in descending to the point M′. To effect this integration, we must eliminate one of the variables contained in the second member, by means of the relations

$$ds=\sqrt{(dx^2+dy^2)}\,\ldots\ldots\,(226),$$

$$y^2=2ax-x^2\,\ldots\ldots\,(227).$$

The latter being differentiated, gives

$$ydy=(a-x)dx;$$

and consequently,

$$dy^2=\frac{(a-x)^2}{y^2}dx^2.$$

This value being substituted in equation (226), we find

$$ds=\sqrt{\left[\left(1+\frac{(a-x)^2}{y^2}\right)dx^2\right]}=dx\sqrt{\left(\frac{y^2+(a-x)^2}{y^2}\right)};$$

or, replacing y^2 by its value given in equation (227), and reducing, we obtain

$$ds = dx\sqrt{\frac{a^2}{y^2}} = \pm\frac{adx}{y} = \pm\frac{adx}{\sqrt{(2ax-x^2)}} = \pm\frac{adx}{\sqrt{[(2a-x)x]}};$$

whence,

$$dt = -\frac{adx}{\sqrt{[(2a-x)x]}\sqrt{[2g(h+b-x)]}}.$$

The negative sign is here prefixed to the second member, because the co-ordinate x is a decreasing function of the time t.

466. If the initial velocity be supposed equal to zero, we shall have

$$h=0;$$

and if at the same time the arc through which the oscillation is performed be supposed extremely small, we can neglect x in comparison with $2a$, and the value of dt will be then reduced to

$$dt = -\frac{adx}{\sqrt{(2ax)}\sqrt{[2g(b-x)]}}.$$

This equation may be put under the form

$$dt = -\tfrac{1}{2}\sqrt{\frac{a}{g}} \times \frac{dx}{\sqrt{[(b-x)x]}} \; \ldots\ldots \; (228).$$

The value of t will be immediately obtained by an integration of the formula

$$\int\frac{dx}{\sqrt{(bx-x^2)}} \; \ldots\ldots \; (229);$$

which, by a comparison with (219), gives

$$a = \tfrac{1}{2}b, \quad z = x:$$

and by substituting these values in equation (219), which is

$$\int\frac{dz}{\sqrt{(2az-z^2)}} = \text{arc}\left(\text{versed sine} = \frac{z}{a}\right),$$

it becomes

$$\int\frac{dx}{\sqrt{(bx-x^2)}} = \text{arc}\left(\text{versed sine} = \frac{x}{\frac{1}{2}b}\right)$$
$$= \text{arc}\left(\text{versed sine} = \frac{2x}{b}\right).$$

But, in general, the cosine of the arc corresponding to the versed sine c and radius unity, being equal to $1-c$, we shall have

$$\text{arc}\left(\text{versed sine}=\frac{2x}{b}\right)=\text{arc}\left(\cos=1-\frac{2x}{b}\right)$$
$$=\text{arc}\left(\cos=\frac{b-2x}{b}\right).$$

This value of the expression (229) being substituted in equation (228), we shall find

$$t=-\tfrac{1}{2}\sqrt{\frac{a}{g}}\times\text{arc}\left(\cos=\frac{b-2x}{b}\right)+C\ldots\ldots(230).$$

467. The constant may be determined from the consideration that when $t=0$, $x=b$; these values reduce the equation (230) to

$$0=-\tfrac{1}{2}\sqrt{\frac{a}{g}}\times\text{arc}\,(\cos=-1)+C.$$

If π denote the semi-circumference of a circle whose radius is unity, we shall have (*Fig.* 174)

$$\text{arc}\,(\cos=-1)=\text{arc BCA}=\pi;$$

and consequently,

$$C=\tfrac{1}{2}\sqrt{\frac{a}{g}}\,.\,\pi.$$

By substituting this value in the equation (230), we obtain

$$t=\tfrac{1}{2}\sqrt{\frac{a}{g}}\left[\pi-\text{arc}\left(\cos=1-\frac{2x}{b}\right)\right]\ldots\ldots(231).$$

The integral being taken between the limits $x=b$, which corresponds to $t=0$, and x equal to any assumed absciss, will make known the time of descent from the point M (*Fig.* 182) to the point M′ corresponding to the assumed value of x.

468. When we wish to obtain the time of descent to the lowest point E, we make $x=0$, in the preceding expression; and since the arc whose cosine is unity is equal to zero, we shall have

$$t=\tfrac{1}{2}\pi\sqrt{\frac{a}{g}}\ldots\ldots(232).$$

469. When the material point arrives at the point E, it

will have acquired its maximum velocity; for the velocity being expressed by

$$v=\sqrt{(2gz)},$$

it will evidently be a maximum at that point of which the ordinate z is the greatest. Thus, in virtue of the velocity acquired at E, the moveable point will describe the arc EN; and since this arc changes its sign in passing through zero, we find for the expression of the time requisite for the point to arrive at N′

$$t=\tfrac{1}{2}\sqrt{\frac{a}{g}}\left[\pi+\text{arc}\left(\cos=1-\frac{2x}{b}\right)\right]\ldots\ldots(233).$$

If from this expression we subtract that given by (232), which expresses the time of descent from the point M to the point E, there will remain

$$\tfrac{1}{2}\sqrt{\frac{a}{g}}\times\text{arc}\left(\cos=1-\frac{2x}{b}\right),$$

an expression for the time of ascent from E to N′: this time is equal to that employed in descending from M′ to E, as may be proved by taking the difference between equations (231) and (232).

Finally, when the material point shall have arrived at the point N, situated in the horizontal line passing through M, we shall have $x=b$, and the expression arc $\left(\cos=1-\frac{2x}{b}\right)$ will then become arc $(\cos=-1)=\pi$; thus, the equation (233) will be reduced to

$$t=\tfrac{1}{2}\sqrt{\frac{a}{g}}\times 2\pi.$$

Such will be the value of the time required by the moveable point to describe the whole arc MEN. This time being denoted by T, we have

$$\text{T}=\pi\sqrt{\frac{a}{g}}\ldots\ldots(234).$$

The velocity of the material point upon its arrival at the point N will be equal to zero; for, since the initial velocity was supposed equal to zero, we have

$$h=0;$$

and this value, taken in connexion with that of $x=b$, reduces the equation $v=\sqrt{[2g(h+b-x)]}$ to

$$v=0.$$

The motion of the material point being entirely destroyed when it arrives at the point N, the force of gravity will cause it again to descend, and since the circumstances of the motion are precisely similar to those presented when the point commenced its motion at M, a second oscillation will be performed in the same time, and a similar motion will continue indefinitely.

470. The equation (234) being independent of the quantity b which expresses the vertical distance MK, it follows that if the point of departure had been taken at M′, instead of at M, the time of oscillation would have been the same; and consequently, that if several material points depart from the different points M, M′, M″, &c., they will all perform their oscillations in the same time. It should be recollected, however, that this result has only been obtained on the supposition that the arcs described are extremely small.

471. These oscillations of equal duration are called *isochronal*. But if the length of the pendulum be supposed variable, the time of vibration will likewise vary: for, if l and l' represent the lengths of two pendulums, whose oscillations are performed in the times T and T′, we shall have

$$T=\pi\sqrt{\frac{l}{g}},\ T'=\pi\sqrt{\frac{l'}{g}};$$

hence,

$$T : T' :: \sqrt{l} : \sqrt{l'} \ldots\ldots (235).$$

Thus, if the time of oscillation T of one pendulum be accurately known, we can determine by the preceding proportion the length l' of a pendulum which shall vibrate in an arbitrary time T′.

472. To ascertain with greater precision the time of a single oscillation, we will represent by N the number of oscillations made by the pendulum whose length is l in a time t, and by N′ the number of oscillations of the pendulum l' in the same time t: we shall then have

$$T=\frac{t}{N}, \text{ and } T'=\frac{t}{N'} \; \ldots\ldots \; (236).$$

By means of these values, the proportion (235) is reduced to

$$N'^2 : N^2 :: l : l',$$

whence,

$$l'=\frac{lN^2}{N'^2}.$$

When the number of oscillations made by a pendulum of a given length, in a given time, has been ascertained from observation, we can calculate the length of the pendulum which will oscillate in a second of time.

If an error be committed in observing the time t, this error will be greatly reduced by being divided by the number of oscillations, and if this number be taken large, the effect of the error upon the time of a single vibration may be regarded as insensible.

473. It is on this principle that the length of the seconds pendulum, which makes 86,400 oscillations in a mean solar day, in vacuo, and at the latitude of New-York, has been found equal to

$$39\overset{\text{in.}}{.}10168=3\overset{\text{ft.}}{.}25847, \text{ nearly.}$$

474. To determine the value of g, the measure of the intensity of the force of gravity, we employ the equation (234), which gives

$$g=\frac{\pi^2 l}{T^2};$$

and by making in this equation

$$T=1'', \quad l=39\overset{\text{in.}}{.}10168, \quad \text{and } \pi=3\overset{\text{ft.}}{.}1415926,$$

or,

$$\pi^2=9.8696046;$$

we find

$$g=385\overset{\text{in.}}{.}9183=32\overset{\text{ft.}}{.}1598.$$

475. If g and g' represent the intensities of gravity at different places, and l and l' the lengths of two pendulums which oscillate in the times T and T', we shall have

$$T=\pi\sqrt{\frac{l}{g}}, \quad T'=\pi\sqrt{\frac{l'}{g'}};$$

from which we deduce

$$T : T' :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l'}{g'}} \ldots . (237).$$

Let N and N′ represent the numbers of oscillations made by these pendulums in the time t; T and T′ will be given in functions of t by equations (236), and their values being substituted in the proportion (237), will give, after reduction,

$$\frac{1}{N} : \frac{1}{N'} :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l'}{g'}}.$$

If the same pendulum be used at the two places, l and l' will be equal to each other, and the preceding proportion will become

$$\frac{1}{N} : \frac{1}{N'} :: \sqrt{\frac{l}{g}} : \sqrt{\frac{l}{g'}};$$

whence,

$$g' = \frac{gN'^2}{N^2}$$

Of the Centrifugal Force.

476. If a material point be supposed to move around a fixed centre C, describing the curve LMK (*Fig.* 183), and if, upon its arrival at the point L, the connexion with the centre be suddenly destroyed, the material point will, in virtue of the law of inertia, continue to move in the direction of the tangent LT. But if we conceive the point to be compelled to describe the curve, it will leave the tangent, and will after a certain time arrive at the point M. The arc LM being supposed indefinitely small, the angle LCM will be so likewise, and the lines LC and MC may be considered as parallel. Thus, replacing CM by the parallel C′M, and constructing the parallelogram LDMN, it appears that the material point, if free, would describe the side LD, while by its connexion with the fixed centre it is caused to describe the diagonal LM; the effect of the force which draws the point towards the centre has therefore been to move it through the space MD.

The point may be supposed to be retained on the curve LMK, either in virtue of a force of attraction which is constantly directed towards the centre C, or by the resistance opposed by the curve regarded as material; or, finally, by being connected with the point C, by means of a cord of variable length.

Whilst the point is describing the elementary arc LM, we can regard it as moving upon the equal arc of the osculatory circle, and can suppose it to be retained on this arc by means of a thread of an invariable length, attached to the centre of the osculatory circle. Moreover, since this thread will experience a tension only in consequence of the resistance offered by the material point to the force which tends to deflect it from the tangent, this tension or the resistance opposed by the point will be precisely equal to the force which causes it to deviate from the tangent. This resistance is exerted in the direction of the radius of curvature, and its constant tendency is to remove the material point from the centre of curvature. Hence, it is called the *centrifugal force*; and the force which constantly urges a body towards any fixed centre is called a *centripetal force.*

The centrifugal force evidently corresponds to the quantity represented by $\frac{v^2}{\gamma}$ in Arts. 451 and 452.

477. To determine directly the expression for the centrifugal force, we replace the infinitely small arc LM by the chord of the osculatory circle at the point L (*Fig.* 184). Then, the versed sine LN will represent the space through which the point would be drawn in virtue of the action of the centrifugal force, during the time occupied by the point in describing the arc LM. From the known property of the circle, we have

$$\mathrm{LN} : \mathrm{LM} :: \mathrm{LM} : \mathrm{LE};$$

or, by substituting the arc for its equal the chord,

$$\mathrm{LN} : ds :: ds : 2\gamma :$$

hence,

$$\mathrm{LN} = \frac{ds^2}{2\gamma};$$

and by substituting for ds its value vdt, we find

$$\mathrm{LN} = \frac{v^2 dt^2}{2\gamma} \quad \ldots\ldots (238).$$

A second expression for the value of LN may be obtained in the following manner. The time required to describe the arc LM will be represented by dt, since this arc is itself denoted by ds; hence, dt will likewise represent the time in which the material point would be caused to describe a space equal to LN under the influence of the centrifugal force alone. Moreover, the centrifugal force acts incessantly, and during the infinitely short time dt, its intensity may be considered invariable. If, therefore, we regard this force as constant, and denote its intensity by f, the circumstances of motion of the point, under the influence of this force, will be expressed by the equations

$$\frac{dv}{dt}=f, \quad \frac{ds}{dt}=v;$$

and by integration,

$$v=t.f, \quad s=\tfrac{1}{2}t^2.f.$$

But the space LN being that which corresponds to the time dt, if in the preceding equations we make $\text{LN}=s$, t will become dt; we shall thus have

$$\text{LN}=\tfrac{1}{2}dt^2\times f.$$

This value of LN being substituted in equation (238) gives, after reduction,

$$f=\frac{v^2}{\gamma}.$$

478. If the material point be supposed to have a circular motion,—as, for example, when a stone is whirled round in a sling, γ will become the radius of the circle described, and the expression for the centrifugal force will then be

$$f=\frac{v^2}{\text{R}} \; \ldots\ldots \; (239).$$

Let h represent the height due to the velocity v; the following relation will then subsist (Art. 401),

$$v^2=2gh;$$

eliminating v^2 between this equation and that which precedes, we obtain

$$\frac{f}{g}=\frac{2h}{\text{R}};$$

from which we conclude, that *the centrifugal force is to the force of gravity, as twice the height due to the velocity is to the radius of the circle described by the material point.*

479. If a semicircle EAF (*Fig.* 185) be supposed to revolve about its diameter EF=2R, the point A, the middle of arc EAF, will describe a circumference equal to $2\pi R$; if this motion be performed uniformly in the time T, with the velocity v, we shall have the relation

$$v \times T = 2\pi R;$$

and by eliminating v between this equation and (239), we find

$$f = \frac{4\pi^2 R}{T^2} \ldots\ldots (240).$$

In like manner, if f' represent the centrifugal force of a point which describes uniformly the circumference of a circle whose radius is R′, in the time T′, we shall have

$$f' = \frac{4\pi^2 R'}{T'^2};$$

and consequently,

$$f : f' :: \frac{R}{T^2} : \frac{R'}{T'^2} \ldots\ldots (241).$$

From this proportion we immediately conclude, that *when the radii* R *and* R′ *are equal, the centrifugal forces will be in the inverse ratio of the squares of the times of revolution; and that when the times are equal, the forces will be directly as the radii.*

480. The effect of the centrifugal force at the equator, caused by the revolution of the earth upon its axis, can now be estimated. For, the equatorial radius of the earth being 20920300 feet, we replace R by this value in equation (240), substituting at the same time the values of π and T. But we have, approximatively,

$$\pi = 3.1415926, \quad \pi^2 = 9.8696046.$$

The time T is determined from the consideration that the earth performs a revolution upon its axis in 0.997269 days, the day being composed of 86400 seconds. Thus we shall have

$$T = 0.997269 \times 86400'' = 86164''.$$

Substituting this value and that of R in equation (240), there results

$$f=0\overset{\text{ft}}{.}1112 \; \ldots\ldots \; (242).$$

481. Having found the value of f, we can determine the intensity G of the force of gravity which would be observed at the equator if the earth were immoveable. For, since the force f is directly opposed to the force G, a portion of the latter will be destroyed by f; and hence, if g denote the intensity of gravity as determined by observation, we shall have

$$g=G-f;$$

or,

$$G=g+f:$$

substituting in this expression the value of f given by equation (242), and that of g, which at the equator is 32.0861 ft., we find

$$G=32\overset{\text{ft}}{.}0861+0\overset{\text{ft}}{.}1112=32\overset{\text{ft}}{.}1973 \; \ldots\ldots \; (243).$$

To determine the relation between the centrifugal force and the force of gravity, we divide equation (242) by equation (243), which gives

$$\frac{f}{G}=\frac{0.1112^{\text{ft}}}{32.1973^{\text{ft}}}=\frac{1}{289} \text{ nearly} \; \ldots\ldots \; (244).$$

482. The proportion (241) will furnish a solution to the following problem:

To find the time in which a revolution of the earth should be performed, in order that the centrifugal force at the equator may be equal to the force of gravity.

Let T' represent the required time of revolution, and f' the corresponding centrifugal force; we shall then have, by the nature of the problem,

$$f'=G, \text{ and } R'=R;$$

these values substituted in the proportion (241) reduce it to

$$f : G :: \frac{1}{T^2} : \frac{1}{T'^2}:$$

whence we obtain

$$T'^2=\frac{f}{G}.T^2.$$

If the fraction $\frac{f}{G}$ be now replaced by its value (244), we shall find

$$T'=\frac{T}{\sqrt{289}}=\frac{T}{17}.$$

Thus, *if the earth's rotation were seventeen times more rapid than it actually is, the centrifugal force at the equator would be equal to the gravity.*

483. To find the diminution of the gravity produced by the centrifugal force at any other point on the earth's surface, it will be necessary to determine the effect of the centrifugal force in the direction of the vertical BZ (*Fig.* 185) drawn through the point under consideration.

For this purpose, we will regard the earth as spherical, it being nearly so: the latitude of the point B being then represented by the arc AB, it will be measured by the angle

$$BOA=ZBC=\psi.$$

Denoting by R the radius AO of the earth, and by R′ the radius BD of the parallel of latitude passing through B, we shall have

$$R'=R\cos OBD;$$

or,

$$R'=R\cos\psi.$$

Let the centrifugal force at the point B, which is exerted in the direction of the radius DB, be represented by the line BC, and resolve it into the two components B*b* and B*c*. The force BC will, by Art. 479, be expressed by $\frac{4\pi^2R'}{T^2}$, and the component f' in the direction of the vertical BZ, which is represented by B*b*, will be given by the relation

$$f'=\frac{4\pi^2R'}{T^2}\times\cos\psi:$$

and by substituting in this relation the value of R′, we shall obtain

$$f'=\frac{4\pi^2R}{T^2}\times\cos^2\psi.$$

The factor $\frac{4\pi^2R}{T^2}$ represents the centrifugal force f at the equator; this equation may therefore be transformed into the proportion

$$f : f' :: 1 : \cos^2 \psi;$$

from which we conclude, that *the diminutions of gravity at different places on the earth's surface, arising from the action of the centrifugal force, are proportional to the squares of the cosines of the latitudes.*

484. The latitude of New-York being 40° 42′ 40″, its cosine will be

$$0.7580;$$

and by multiplying the value of f (242) by the square of this number, or by

$$0.5746,$$

we find

$$f' = 0\overset{\text{ft.}}{.}0639.$$

If G′ represent the value of the force of attraction, or that which the observed gravity would have in the latitude of New-York, if the earth were immoveable, the gravity actually observed being denoted by g', we shall have, as in Art. 481,

$$G' = g' + f'.$$

The observed gravity g', in the latitude of New-York, being

$$32\overset{\text{ft.}}{.}1598,$$

we find, by substituting this value and that of f' in the preceding equation,

$$G' = 32\overset{\text{ft.}}{.}1598 + 0\overset{\text{ft.}}{.}0639 = 32\overset{\text{ft.}}{.}2237 \; \ldots\ldots \; (245).$$

Of the System of the World.

485. In discussing the properties of the centre of gravity, we have already had occasion to consider that remarkable force exerted by the earth, in virtue of which all bodies are solicited in directions perpendicular to its surface. The existence of this force was not entirely unknown to the ancients: Anaxagoras, and his disciples, Democritus, Plutarch, Epicurus, and others, admitted the existence of such a force; and similar opinions were entertained by Kepler, Galileo, Huygens, Fermat, Roberval, &c., in modern times. The celebrated

Kepler distinctly affirms, in his work *De Stella Martis*, that the force of attraction is not confined to bodies situated upon the surface of the earth, but that it extends to the most distant stars.

This bold conception remained long unimproved, from the difficulty of verifying its truth. The effects of gravity at the earth's surface were measured by Galileo.

Lord Bacon, suspecting that the intensity of this force must vary with the distance from the centre of the earth, endeavoured to verify the truth of this conjecture by observing the distances through which bodies would fall, in a given time, at different elevations above the surface of the earth. But, however great were these elevations, they proved too small to render the variations in the intensity of gravity perceptible.

Newton extended his views yet further; and not satisfied with the mere conjecture that the intensity of gravity was subject to variation, he endeavoured to measure the law of its diminution. He adopted, as the most probable law of diminution, that of the inverse ratio of the square of the distance; such being the law according to which light and other emanations were known to be propagated. To test the truth of this supposition, he endeavoured to obtain a measure of the intensity of gravity at the distance of the moon, and the only obstacle to this determination arose from an imperfect knowledge of the moon's distance, and of the dimensions of the earth; but more exact determinations of these elements having been supplied by Picard and others, he was enabled to base his calculations on more accurate data.

486. The first element to be determined in this investigation, is the intensity of gravity at the surface of the earth. The method of obtaining this quantity by the oscillations of a pendulum has already been explained in Arts. 474 and 484: it was thus found, that in the latitude of New-York, and on the supposition that the earth was immoveable,

$$G'=32\overset{ft.}{.}2237 \ldots\ldots (246).$$

This quantity is nearly the same for all places on the surface of the earth.

To ascertain the diminution which the intensity of gravity should sustain at the distance of the moon, according to the supposed law of Newton, it will be necessary to know the distance of the moon from the centre of the earth. This distance depends on the horizontal parallax of the moon.

487. Let CL and HL (*Fig.* 186) represent two lines drawn from the moon to the two extremities of the terrestrial radius, the line HL being perpendicular to this radius. The angle HLC is called the horizontal parallax of the moon, and its mean value, according to Delambre, is 57′. If therefore, the radius of the earth be taken as unity, we shall have

$$\mathrm{CL}\sin \mathrm{L}=\mathrm{CH}=1,$$

and consequently,

$$\mathrm{CL}=\frac{1}{\sin 57'}=60.314;$$

this value differs but little from that employed by Newton, who supposed the mean distance to be 60¼.

488. If we denote by γ the intensity of gravity at the distance of the moon, upon the hypothesis that it decreases according to the law of the inverse ratio of the square of the distance, and by s' the space which it would cause a body to describe in the time t, we shall have

$$\mathrm{G}' : \gamma :: (60.314)^2 : 1^2;$$

whence,

$$\gamma=\frac{\mathrm{G}'}{(60.314)^2}.$$

Such is the expression for the velocity which should be imparted by gravity, at the distance of the moon, in a second of time, if the hypothesis assumed be correct.

489. By substituting this value for g in the general formula,

$$s=\tfrac{1}{2}gt^2,$$

and replacing s by s', we shall obtain the space described in the time t.

Thus, if we suppose the time to be one minute, or 60″, we shall have for the space s', which the body would describe in a minute of time,

$$s'=\frac{\frac{1}{2}(60)^2\mathrm{G}'}{(60.314)^2}\ \ldots\ldots\ (247).$$

490. If we neglect the decimal fraction, in the denominator, the equation reduces to

$$s' = \tfrac{1}{2}G';$$

from which we conclude that the space described by a body moving from rest, in a *minute* of time, at the distance of the moon, should be equal, according to Newton's hypothesis, to the space passed over in a *second* of time, at the surface of the earth.

But if we take account of the decimal fraction, the equation (247) will give by reduction,

$$s' = \tfrac{1}{2}G' \times 0.9896;$$

and by substituting the value of G′ (246), we have

$$s' = \tfrac{1}{2} \times \overset{\text{ft.}}{32.2237} \times 0.9896;$$

or, by performing the multiplications indicated,

$$s' = \overset{\text{ft.}}{15.9443} \; . \; . \; . \; . \; . \; (248).$$

Such would be the distance described by the body in a minute of time, at the distance of the moon, if the body were supposed to move from a state of rest.

491. Let us now examine whether this result is confirmed by experience. For this purpose, let the moon when at its mean distance be supposed to describe the arc LM (*Fig.* 187) in a minute of time: if the lines LQ and QM be drawn respectively parallel to the sine and versed sine of the arc LM, we may regard LM as the diagonal of a parallelogram of which LQ and LP will be the sides. If the moon were not solicited by the earth's attraction, it would describe the tangent LQ, and if solicited by this attraction solely, it would describe the line LP in the same time: this line LP will therefore serve to determine the intensity of the earth's attraction, and it will evidently be equal to the versed sine of the angle LCM.

But since the mean radius r of the moon's orbit undergoes but a very slight variation in a minute of time, this portion of the orbit may be regarded as the arc of a circle described with the radius r; and since the moon, when at its mean distance, moves with nearly its mean velocity, we shall have, by calling T the time of a sidereal revolution, or the time required by the moon to return to the same point of the heavens,

$$T : 1 \text{ minute} :: 360^\circ : \text{angle LCM};$$

whence,

$$\text{angle LCM} = \frac{360^\circ}{T}.$$

This time of revolution being known from observation to be 27 days 7 hours 43 minutes, or 39343′, we will replace T by this value, at the same time reducing the degrees to seconds, to render the division possible; we thus obtain

$$\text{angle LCM} = \frac{1296000''}{39343} = 32''.94.$$

492. The question is thus reduced to finding the versed sine of an arc of 32″.94, in a circle described with the mean radius of the lunar orbit.

To effect this, let the perpendicular CI (*Fig.* 187) be drawn to the middle of the chord LM: the right-angled triangles LMP, LCI, having the common angle L, will be similar, and will give the proportion

$$LC : IL :: LM : LP;$$

or,

$$LC : IL :: 2IL : LP:$$

whence,

$$LP = \frac{2IL^2}{LC} \; \ldots\ldots \; (249).$$

Let θ represent the angle LCI equal to $\frac{1}{2}$LCM, and r the mean radius LC, we shall have

$$IL = r \cdot \sin\theta,$$

and the equation (249) will become

$$LP = 2r \cdot \sin^2\theta;$$

or, by substituting the value of the angle θ,

$$LP = 2r \cdot \sin^2 16''.47 \; \ldots\ldots \; (250).$$

If a denote the mean radius of the earth, the mean radius of the lunar orbit will be expressed by

$$r = (60.314)a.$$

493. But the mean radius of the earth determined by the measurement of a degree upon its surface, being equal to

20886500 feet,

we shall have, by substitution,

$$LP = 60.314 \times 2 \times \overset{\text{ft.}}{20886500} \times \sin^2 (16''.47),$$

and changing

$$\frac{\text{sine whose radius is unity}}{1} \text{ into } \frac{\text{tabular sine}}{\text{tabular radius}},$$

for the purpose of using logarithms, we find

Log 60.314 - - - - - - - - - - -	1.7804181
Log 2 - - - - - - - - - - - - -	0.3010300
Log 20886500 - - - - - - - - -	7.3198657
Log sin² (16″.47), or 2.log sin (16″.47)	11.8041388
Corresponding number = 16.0492 ft. -	1.2054526

494. It thus appears that the moon falls towards the earth in a minute of time a distance of 16.0492 ft., corresponding very nearly with that deduced on the supposition that the intensity of gravity varies inversely as the square of the distance from the centre of the earth. The difference between the two results amounts only to about 0.15 ft. in the space fallen through by the moon in a minute of time, and will consequently become nearly insensible in the space described in one second. Moreover, this slight difference might fairly have been anticipated, since mean values of the several quantities which enter into the calculation have alone been employed.

495. The remarkable accordance exhibited by the preceding calculation between the results of theory and experience, justifies us in concluding that the force of gravity exerts an influence at the distance of the moon, but that its intensity is less than at the surface of the earth, in the inverse ratio of the squares of the distances from the centre of the earth. The truth of this supposition has been uniformly confirmed by experience; astronomical tables calculated upon the hypothesis of Newton assign the positions of the celestial bodies such as they are determined by direct observation,

without presenting a single exception. This hypothesis may therefore be regarded as fully established by experience. Those general truths which are designated *the laws of Kepler*, and which have been repeatedly verified by observation, serve to establish the hypothesis of Newton in the most clear and decisive manner. These laws may be enunciated as follows:

1°. *The planets describe ellipses, having the centre of the sun at one of their foci.*

2°. *The areas of the elliptical sectors described by the radius vector drawn from the planet to the centre of the sun are constantly proportional to the times of description.*[*]

3°. *The squares of the times of revolution of the several planets are proportional to the cubes of their mean distances from the sun.*

496. The first of these laws, as will be demonstrated, is a particular case resulting from the more general law of nature, which requires that a body subjected to the action of a force which varies inversely as the square of the distance from a fixed point, should necessarily describe a conic section.

The second law has already been noticed (Art. 435), and subsists in general for every body which is constantly attracted towards a fixed point. The question is thus reduced to proving the truth of the first and third of Kepler's laws, after adopting the hypothesis of Newton.

---

[*] In the general course of reasoning which is here applied to the motions of the planets, these bodies are regarded as mere material points. The propriety of making this supposition will not fully appear until after we have discussed the circumstances of motion of a solid body whose several particles are acted upon by incessant forces. It will then be found that the motion of the centre of gravity of such a body will be precisely the same as though the mass of the body were concentrated at its centre of gravity, and the several forces applied directly to that point. Thus the case will be reduced to that of the motion of a material point. It is, however, quite obvious that this hypothesis cannot differ much from the truth; for, since the dimensions of the planets are exceedingly minute when compared with their distances from the sun, it follows that every particle in the planet will be acted upon by a force which is very nearly equal and parallel to the force exerted upon that particle which coincides with the centre of gravity of the planet. Thus, the particles, being acted upon by parallel and equal forces, will have the same motions as though they were unconnected with each other; and the reasoning may be applied to any one of these particles, the central one, for example.

497. Let the origin of co-ordinates be placed at the centre of attraction (*Fig.* 188), which corresponds to the centre of the sun, for the planetary system, and let R denote the value of the force of attraction exerted by the sun upon one of the planets, and r the radius vector drawn to the planet.

The force R coinciding in direction with the radius vector mA, if we represent by φ the angle mAP, which the radius vector forms with the axis of x, the components of the force R in the directions of the axes will be

$$\mathrm{X}=\mathrm{R}\cos\varphi,\quad \mathrm{Y}=\mathrm{R}\sin\varphi.$$

But in the right-angled triangle AmP, we have

$$\cos\varphi=\frac{\mathrm{AP}}{m\mathrm{A}}=\frac{x}{r},\quad \sin\varphi=\frac{m\mathrm{P}}{m\mathrm{A}}=\frac{y}{r}.$$

Thus, the components X and Y of the force R will be expressed by

$$\mathrm{X}=\mathrm{R}\frac{x}{r},\quad \mathrm{Y}=\mathrm{R}\frac{y}{r}:$$

and since the incessant force is supposed to act in the direction from m towards A, it will tend to diminish the co-ordinates AP$=x$, and P$m=y$, of the point m: hence, the components of the incessant force should be affected with the negative sign (Art. 51); the two preceding equations will thus become

$$\mathrm{X}=-\mathrm{R}\frac{x}{r},\quad \mathrm{Y}=-\mathrm{R}\frac{y}{r};$$

or, replacing X and Y by their values given in equations (180), we obtain

$$\frac{d^2x}{dt^2}=-\mathrm{R}\frac{x}{r},\quad \frac{d^2y}{dt^2}=-\mathrm{R}\frac{y}{r}\;\ldots\ldots(251).$$

498. For the purpose of integrating these equations, let the first be multiplied by y, and the second by x: taking the difference of the products, and multiplying by dt, there will result

$$y\frac{d^2x}{dt}-x\frac{d^2y}{dt}=0;$$

the integral of which is

$$\frac{ydx-xdy}{dt}=a \;.\;.\;.\;.\;. \; (252),$$

the arbitrary constant introduced by integration being denoted by a.

499. To obtain a second integral, we multiply the first of equations (251) by $2dx$, and the second by $2dy$, and take their sum; we thus obtain,

$$\frac{2dx\,.\,d^2x+2dy\,.\,d^2y}{dt^2}=-2\mathrm{R}\left(\frac{xdx+ydy}{r}\right) \;.\;.\;.\;.\;. \; (253).$$

The second member of this equation containing the three variables x, y, and r, we eliminate two of them by means of the relation $x^2+y^2=r^2$, which gives, by differentiation,

$$xdx+ydy=rdr\,;$$

this value substituted in the second member of equation (253) reduces it to

$$\frac{2dx\,.\,d^2x+2dy\,.\,d^2y}{dt^2}=-2\mathrm{R}dr\,;$$

or, since dt is regarded as constant,

$$d\left(\frac{dx^2+dy^2}{dt^2}\right)=-2\mathrm{R}dr.$$

Integrating, and denoting by b the arbitrary constant, we obtain

$$\frac{dx^2+dy^2}{dt^2}=b-2\int\mathrm{R}dr \;.\;.\;.\;.\;. \; (254).$$

500. The quantity $\mathrm{R}dr$ is affected with the sign of integration, the intensity of the force R being supposed a function of the distance r; the nature of this function will remain arbitrary, so long as we do not adopt a particular hypothesis.

501. This equation still containing three variables, we reduce the number to two (φ and r), by introducing the values of x and y, expressed in functions of r, and the angle φ included between the radius vector and the axis of x; these values are given by the formulas,

$$x=r\,.\cos\varphi, \quad y=r\,.\sin\varphi \;.\;.\;.\;.\;. \; (255).$$

By differentiating, we have,

$$\left.\begin{array}{l} dx=-r\sin\varphi d\varphi+\cos\varphi dr \\ dy=r\cos\varphi d\varphi+\sin\varphi dr \end{array}\right\} \;.\;.\;.\;.\;. \; (256)\,;$$

and the values of x, y, dx, and dy given by equations (255) and (256), being substituted in equation (252), transform it into

$$-r^2\frac{d\varphi}{dt}=a \,.\,.\,.\,.\,.\, (257).$$

The sum of the squares of equations (256) gives, after reduction,

$$dx^2+dy^2=r^2d\varphi^2+dr^2 \,.\,.\,.\,.\,.\, (258)$$

and by substituting this value in equation (254), we obtain,

$$\frac{r^2d\varphi^2+dr^2}{dt^2}=b-2\int\mathrm{R}dr \,.\,.\,.\,.\,.\, (259).$$

502. To determine the equation of the curve described by the moveable point, we eliminate dt between equations (257) and (259): the first of these gives

$$dt=-\frac{r^2d\varphi}{a};$$

this value, introduced into the second, transforms it into

$$\frac{a^2r^2d\varphi^2+a^2dr^2}{r^4d\varphi^2}=b-2\int\mathrm{R}dr;$$

whence we deduce,

$$d\varphi=\frac{adr}{r\sqrt{(br^2-a^2-2r^2\int\mathrm{R}dr)}} \,.\,.\,.\,.\,.\, (260).$$

This equation being integrated, and the values of the constants being determined, we shall have a relation between the radius vector r and the angle φ.

503. To determine the constants a and b, we will resume the integrals,

$$\frac{ydx-xdy}{dt}=a, \quad \frac{r^2d\varphi^2+dr^2}{dt^2}=b-2\int\mathrm{R}dr \,.\,.\,.\,.\,.\, (261).$$

The integral of the first of these equations is, by Art. 435,

$$2\,.\,\text{sector LA}m=at \,.\,.\,.\,.\,.\, (262);$$

consequently, by making $t=1$, we shall find that a is equal to twice the sector described in a unit of time.

The same result may be obtained from the equation

$$r^2\frac{d\varphi}{dt}=a \,.\,.\,.\,.\,.\, (263);$$

for $d\varphi$ being the infinitely small arc described in the time dt, by a point on the radius vector whose distance from the centre of attraction is equal to unity, $rd\phi$ will be the arc described in the same time with the radius r; hence $\frac{1}{2}r \,.\, rd\phi$, or $\frac{r^2 d\phi}{2}$, will represent the infinitely small sector described by the radius vector in the time dt; but since the areas described are proportional to the times of description (Art. 435), we can find the area described in the time 1, by the proportion

$$\frac{r^2 d\phi}{2} : dt :: \text{area described in time unity} : 1;$$

whence,

$$\text{area described in time unity} = \frac{r^2 d\phi}{2dt};$$

and consequently, $\frac{r^2 d\phi}{dt}$, or its equal a, will be double the area described by the radius vector in the unit of time.

It may be remarked that the change in the sign of the first member of equation (257) which converts it into (263), is merely equivalent to a change in the position of the fixed line from which the areas described are reckoned: in the first case they are reckoned from the axis of y, and in the second from the axis of x.

From the equation (263) we deduce

$$\frac{d\phi}{dt} = \frac{a}{r^2} \quad \ldots\ldots (264.)$$

The quantity $\frac{d\phi}{dt}$ expresses the angular velocity of the body, or the velocity of that point on the radius vector which is at the distance unity from the centre of attraction; and it appears by the preceding relation, that *the angular velocity varies in the inverse ratio of the square of the radius vector.*

504. From the first of equations (261), we may infer that the quantity a is independent of the law according to which the attractive force is supposed to vary; but the quantity b, which appears in the second equation, will evidently depend on the attractive force, which likewise appears in the same equation. It will therefore be necessary to adopt some hypothesis respecting the law of this force, such, for example,

as the law of Newton, which supposes that different bodies attract each other in the direct ratio of their masses, and the inverse ratio of the squares of their distances.

Let the force exerted by the unit of mass, at the distance k, be denoted by 1; the force exerted by the sun upon a body placed at the distance k will then be expressed by the mass M of the sun, or, in other words, by the number of units which its mass contains: but the mass of a planet attracted by the sun being denoted by m, this planet will exert an attraction upon the sun, which will, for a similar reason, be expressed by its mass m: moreover, since the two forces M and m tend to cause the approach of the two bodies, their effect upon the relative motion of the bodies will be the same as if the force $\mathrm{M}+m$ were concentrated in the sun, and acted on the planet at the distance k. When this distance varies and becomes equal to r, the intensity of the force will likewise vary. Let R denote its intensity at the distance r; the assumed hypothesis will give the proportion

$$\mathrm{M}+m : \mathrm{R} :: \frac{1}{k^2} : \frac{1}{r^2};$$

whence,

$$\mathrm{R}=\frac{k^2(\mathrm{M}+m)}{r^2} \;\ldots\ldots\; (265).$$

Such is the value of the attractive force which, acting at the distance r, will cause the bodies to approach each other.

505. The value thus determined corresponds to that of the incessant force which we have hitherto represented by **R**: we therefore have

$$\int \mathrm{R}dr=\int\frac{k^2(\mathrm{M}+m)}{r^2}dr;$$

putting, for brevity,

$$k^2(\mathrm{M}+m)=\mathrm{M}' \;\ldots\ldots\; (266),$$

the preceding equation will be reduced to

$$\int \mathrm{R}dr=\int\frac{\mathrm{M}'dr}{r^2} \;\ldots\ldots\; (267);$$

but since the quantities M and m and the distance k remain invariable, the quantity M′ will be constant; the equation (267) may therefore be readily integrated, and will give

$$\int R dr = \frac{-M'}{r} + c:$$

replacing $\int R dr$ by this value, and $b-2c$ by b', the equations (259) and (260) will become

$$\frac{r^2 d\varphi^2 + dr^2}{dt^2} = b' + \frac{2M'}{r} \ldots\ldots (268),$$

$$d\varphi = \frac{\alpha dr}{r\sqrt{(b'r^2 - a^2 + 2M'r)}} \ldots\ldots (269).$$

506. To determine the value of the constant b', or its equal $b-2c$, we observe that the equations (258) and (268) give, by comparison,

$$\frac{dx^2 + dy^2}{dt^2} = \frac{r^2 d\varphi^2 + dr^2}{dt^2} = b' + \frac{2M'}{r};$$

and since $\sqrt{(dx^2 + dy^2)}$ is equal to ds, the element of the curve, it appears that the quantity $\frac{r^2 d\varphi^2 + dr^2}{dt^2}$ is equal to $\left(\frac{ds}{dt}\right)^2$, or equal to the square of the velocity estimated in the direction of the tangent to the curve; thus, denoting this velocity by v, the equation (268) will become

$$v^2 = b' + \frac{2M'}{r} \ldots\ldots (270).$$

If V represent the velocity at a given instant, and λ the corresponding value of the radius vector, the equation (270) will contain but a single unknown quantity b', whose value will result

$$b' = V^2 - \frac{2M'}{\lambda}.$$

507. The constant a may also be determined in functions of the initial velocity; for, by replacing $\frac{d\varphi}{dt}$ in the formula

$$v^2 = \frac{r^2 d\varphi^2 + dr^2}{dt^2},$$

by its value $\frac{a}{r^2}$ deduced from equation (264), we shall obtain

$$v^2 = \frac{dr^2}{dt^2} + \frac{a^2}{r^2} \ldots\ldots (271).$$

The quantity dr represents the infinitely small difference ml (*Fig.* 189) between two consecutive radii Am and An; and by regarding the triangle mnl as rectilinear, and right-angled at l, we shall have

$$ml = mn\,.\cos nml,$$

or,

$$dr = ds\,.\cos nml;$$

substituting this value of dr in (271), and changing $\frac{ds}{dt}$ into v, we shall find

$$v^2 = v^2\cos^2 nml + \frac{a^2}{r^2}.$$

But if α denote the value of the angle nml, when v and r are transformed into V and λ, we shall have the relation

$$V^2 = V^2.\cos^2\alpha + \frac{a^2}{\lambda^2};$$

whence,

$$a^2 = \lambda^2 V^2(1 - \cos^2\alpha) = \lambda^2 V^2\sin^2\alpha;$$

and consequently,

$$a = \lambda\,.\,V\,.\sin\alpha.$$

508. Having determined the constants which enter into equation (269), we proceed to integrate it, for the purpose of discovering the nature of the trajectory described by the material point.

To facilitate the integration, make $r = \frac{1}{z}$, and the equation (269) will then become

$$d\varphi = -\frac{adz}{\sqrt{[b' - (a^2z^2 - 2M'z)]}};$$

or,

$$d\varphi = -\frac{adz}{\sqrt{\left[b' + \frac{M'^2}{a^2} - \left(az - \frac{M'}{a}\right)^2\right]}};$$

making

$$az - \frac{M'}{a} = p, \text{ and } b' + \frac{M'^2}{a^2} = A^2,$$

the preceding equation will be reduced to

$$d\varphi=\frac{-dp}{\sqrt{(A^2-p^2)}};$$

and by integrating, we find

$$\varphi+\text{constant}=\text{arc}\left(\cos=\frac{p}{A}\right).$$

Replacing p and A by their values, suppressing the common factor a, and denoting by ψ the arbitrary constant, we obtain

$$\varphi+\psi=\text{arc}\left(\cos=\frac{a^2z-M'}{\sqrt{(a^2b'+M'^2)}}\right);$$

whence,

$$\frac{a^2z-M'}{\sqrt{(a^2b'+M'^2)}}=\cos(\varphi+\psi);$$

and by restoring the value of z in terms of r, we finally obtain

$$a^2-M'r=r\sqrt{(a^2b'+M'^2)}\,.\cos(\varphi+\psi)\,.\,.\,.\,.\,.\,(272).$$

509. The arbitrary constant ψ serves merely to change the direction of the axis with which the radius vector forms the variable angle: if, for example, the angle CAm or φ (*Fig.* 190), formed by the radius vector with the primitive axis AC, be supposed successively equal to 1°, 2°, 3°, &c. and if the variable angle be reckoned from the axis AB, which forms with the axis AC an angle CAB$=\psi$, the angle included between the radius vector Am and the axis AB, will be successively equal to

$$1^\circ+\psi,\quad 2^\circ+\psi,\quad 3^\circ+\psi,\ \&c.;$$

or, in general, to

$$\varphi+\psi.$$

510. The angle $\varphi+\psi$ will disappear from equation (272), when the polar co-ordinates are transformed into rectangular co-ordinates, by means of the formulas

$$r^2=x^2+y^2,\quad x=r\cos(\varphi+\psi),\quad y=r\sin(\varphi+\psi)\,.\,.\,.\,.\,.\,(273);$$

for the first two of these formulas reduce the equation (272) to

$$a^2-M'\sqrt{(x^2+y^2)}=x\sqrt{(a^2b'+M'^2)};$$

which gives, by transposition,

$$M'\sqrt{(x^2+y^2)}=a^2-x\sqrt{(a^2b'+M'^2)}\,.\,.\,.\,.\,.\,(274):$$

squaring and reducing, we find

$$M'^2y^2-b'a^2x^2=a^4-2a^2x\sqrt{(a^2b'+M'^2)} \ldots\ldots (275).$$

This equation appertains to a conic section, or curve of the second degree: it will be the equation of an ellipse or hyperbola, according as b', upon which the sign of the second term depends, is negative or positive; for, in the first case, the terms containing the squares of the co-ordinates will have similar signs, whilst in the second, they will be affected with contrary signs: when b' becomes equal to zero, the term containing x^2 will disappear, and the equation will then appertain to a parabola.

511. If we resolve equation (275) with reference to y, there will result

$$y=\pm\frac{a}{M'}\sqrt{[a^2+b'x^2-2x\sqrt{(a^2b'+M'^2)}]};$$

which proves that every rectangular ordinate is equally divided by the axis of x, and consequently that this axis must necessarily be the greater or lesser axis of the curve: but by introducing into equation (274) the value of the radius vector given by the first of equations (273), we shall obtain

$$M'r=a^2-x\sqrt{(a^2b'+M'^2)};$$

hence it appears that the radius vector is constantly expressed in rational functions of the absciss x, and that the origin therefore corresponds to the focus. Thus the co-ordinate axis of x will coincide with the greater axis of the curve.

512. The second law of Kepler is thus demonstrated to be a consequence of the hypothesis of Newton, and admits of a generalization wholly unknown to its discoverer; he was induced, judging by analogy, to assign *elliptical* orbits to the planets, whereas it appears from the preceding demonstration, that they might have described either *hyperbolas* or *parabolas*. If amongst the comets hitherto observed we have found no examples of a hyperbolic motion, it results from the fact that the chance of a body's describing a curve which shall be sensibly hyperbolic is found to be extremely small. "I have found," says Laplace, "that the chances are at least six thousand to one that a comet which comes within the sphere of the sun's action will describe an extremely elongated ellipse, or a hyperbola, which, by the magnitude of its trans-

verse axis, will be sensibly confounded with a parabola, in that portion of its orbit which can be observed; it is not surprising, therefore, that the hyperbolic motion has not yet been observed."

513. If in equation (275), we make $x=0$, and $y=\frac{1}{2}p$, we shall obtain for the ordinate passing through the focus, or the semi-parameter,

$$\tfrac{1}{2}p=\frac{a^2}{M'}.$$

514. The equation (275) admits of simplification, by making

$$\sqrt{(a^2b'+M'^2)}=n \,.\,.\,.\,.\,.\, (276),$$

and transporting the origin to the centre of the curve: for this purpose we make $x=x'+\alpha$, and dispose of the arbitrary quantity α by the condition that the coefficient of the first power of x' shall vanish. Making these substitutions in equation (275), and dividing by a^2, we find

$$\frac{M'^2}{a^2}y^2-b'x'^2\left.\begin{matrix}-2b'\alpha\\+2n\end{matrix}\right\}x'\left.\begin{matrix}-b'\alpha^2\\+2n\alpha\\-a^2\end{matrix}\right\}=0\,.\,.\,.\,.\,.\,(277).$$

Putting the coefficient of x' equal to zero, we have

$$\alpha=\frac{n}{b'};$$

this value being introduced into the last term of equation (277) reduces it to

$$\frac{n^2}{b'}-a^2.$$

But the equation (276) gives

$$\frac{n^2}{b'}-a^2=\frac{M'^2}{b'};$$

substituting this value for the last term of equation (277), and suppressing the second term, which by hypothesis is equal to zero, we shall obtain

$$\frac{M'^2}{a^2}y^2-b'x'^2+\frac{M'^2}{b'}=0;$$

or by clearing the denominators,

$$b'M'^2y^2-b'^2a^2x'^2+a^2M'^2=0\,.\,.\,.\,.\,.\,(278).$$

In this equation the origin of co-ordinates is at the centre of the curve; hence, if we make $y=0$, and deduce the corresponding value of x', we shall have

$$\text{semi-axis major}=\frac{M'}{b'}\,.\,.\,.\,.\,.\;(279);$$

and by making a similar supposition with respect to x', we find

$$\text{semi-axis minor}=\sqrt{-\frac{a^2}{b'}}.$$

This value becomes imaginary when b' is positive, agreeing with the result in Art. 510, since the curve described is then a hyperbola; but the value is real when b' is negative, the curve then being an ellipse. In this case, if we replace b' by $-b'$, we shall have

$$\text{semi-axis minor}=\frac{a}{\sqrt{b'}}\,.\,.\,.\,.\,.\;(280).$$

515. This result corresponds with that which would have been obtained from the consideration that the minor axis is a mean proportional between the major axis and the parameter, the values of which have been already obtained.

516. Having determined the principal elements of the curve described, it will now be easy to establish the third of Kepler's laws. Let π denote the number 3.1416; then, the area of an ellipse whose semi-axes are represented by A and B will be expressed by πAB; and if A and B be replaced by their values determined in equations (279) and (280), we shall find

$$\text{area of the ellipse described by the planet}=\frac{\pi a M'}{b'\sqrt{b'}}\,.\,.\,.\,.\;(281);$$

or,

$$\text{area of the ellipse described by the planet}=\frac{\pi a}{\sqrt{M'}}\left(\frac{M'}{b'}\right)^{\frac{3}{2}}.$$

But it has already been shown that if t represent the time required by a planet to describe the sector LAm (*Fig.* 188), the equation (262) will give

$$t=\frac{2\text{ sector LA}m}{a}.$$

When t becomes the time of an entire revolution, which we will represent by T, the sector LAm will become the area of the ellipse, and we shall then have

$$\mathrm{T}=\frac{2\pi}{\sqrt{\mathrm{M}'}}\left(\frac{\mathrm{M}'}{b'}\right)^{\frac{3}{2}};$$

and since $\frac{\mathrm{M}'}{b'}$ represents the semi-axis major, we shall have, by representing its value by D,

$$\mathrm{T}=\frac{2\pi}{\sqrt{\mathrm{M}'}}\mathrm{D}^{\frac{3}{2}}:$$

or, replacing M′ by its value (266), we obtain

$$\mathrm{T}=\frac{2\pi\mathrm{D}^{\frac{3}{2}}}{k\sqrt{(\mathrm{M}+m)}}\ \ldots\ldots\ (282).$$

In like manner, for a second planet m', which performs its revolution in the time T′, in an ellipse whose semi-axis major is denoted by D′, we shall have, since the mass of the sun remains invariable,

$$\mathrm{T}'=\frac{2\pi\mathrm{D}'^{\frac{3}{2}}}{k\sqrt{(\mathrm{M}+m')}}\ \ldots\ldots\ (283):$$

but the masses of the planets being extremely small when compared with the mass of the sun, we may neglect the quantities m and m' in comparison with M; and the equations (282) and (283), being then compared, will give the proportion

$$\mathrm{T}:\mathrm{T}'::\mathrm{D}^{\frac{3}{2}}:\mathrm{D}'^{\frac{3}{2}},\quad \text{or } \mathrm{T}^2:\mathrm{T}'^2::\mathrm{D}^3:\mathrm{D}'^3;$$

the squares of the times of revolution will therefore be proportional to the cubes of the greater axes of the orbits described, or to the cubes of the mean distances of the planets from the sun.

517. The inverse problem may also be resolved, and the law of gravitation deduced, from the elliptical motions of the planets. For this purpose, we must adopt the hypothesis that the equation (260) refers to an ellipse: but the polar equation of the ellipse being of the form $\mathrm{C}r\cos\varphi=\mathrm{B}^2-\mathrm{A}r$, its differential will give

$$d\varphi=\frac{\mathrm{B}^2dr}{r\sqrt{[(\mathrm{C}^2-\mathrm{A}^2)r^2-\mathrm{B}^4+2\mathrm{A}\mathrm{B}^2r]}}.$$

The condition of identity between this equation and equation (260) requires that we should have

$$-r\int \mathrm{R}dr = \mathrm{AB}^2 = \text{a constant}, \quad \text{or } -\int \mathrm{R}dr = \frac{\text{constant}}{r};$$

differentiating, and suppressing dr, there remains

$$\mathrm{R} = \frac{\text{constant}}{r^2},$$

which proves that the force varies in the inverse ratio of the square of the distance.

Of the Motions of Projectiles.

518. If an impulse be communicated to a material point in a direction oblique to the surface of the earth, the point being at the same time solicited by the force of gravity, it will describe a trajectory, the nature of which it is proposed to investigate. To determine the circumstances of this motion, we will denote by Ax, Ay, and Az the three co-ordinate axes, the axis Az being supposed vertical. The force of gravity will then tend to diminish the co-ordinates z which are reckoned positive upward, and if its intensity be supposed constant, we shall have

$$\mathrm{X}=0, \quad \mathrm{Y}=0, \quad \mathrm{Z}=-g.$$

These values being substituted in the general equations (180) reduce them to

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=0, \quad \frac{d^2z}{dt^2}=-g;$$

the first two of these equations being multiplied by dt, and integrated, give

$$\frac{dx}{dt}=a, \quad \frac{dy}{dt}=b;$$

the constants a and b represent the velocities of the material point in the directions of the axes of x and y respectively. These velocities distinguish the motion under consideration from that which takes place when the point is projected vertically, their values in the latter case becoming equal to zero.

If the preceding equations be multiplied by dt, and again integrated, we shall obtain

0, and the corresponding

es the range. Thus

and eliminating

This equation app

ated in the plane of

be contained in a ve

519. Since the tra

plane, it will only be

nate axes of x and y

and the latter vertical;

$$\frac{d^2x}{dt^2}$$

Multiplying by dt, and i and

$$\frac{dx}{dt}=a, \quad \frac{dy}{dt}=-gt+b \ldots\ldots (284).$$

If we multiply again by dt, and integrate, we shall obtain

$$x=at+a', \quad y=-\tfrac{1}{2}gt^2+bt+b' \ldots\ldots (285).$$

To determine the constants, we suppose the time to be reckoned from the instant at which the material point leaves the origin of co-ordinates; whence,

$$x=0, \quad y=0, \text{ and } t=0;$$

this supposition gives

$$a'=0, \quad b'=0;$$

and the equations (285) are thus reduced to

$$x=at, \quad y=-\tfrac{1}{2}gt^2+bt.$$

Eliminating t between these two equations, we find

$$y=\frac{b}{a}x-\tfrac{1}{2}g\frac{x^2}{a^2} \ldots\ldots (286).$$

The equations (284) indicate that the constants a and b express the values of the horizontal and vertical components of the velocity at the instant from which the time is reckoned, or when $t=0$. If, therefore, V denote the initial velocity, and α the angle formed by the direction of the initial impulse with the axis of x, the components of this velocity will be as the law of Newton, which supposes that different bodies attract each other in the direct ratio of their masses, and the inverse ratio of the squares of their distances.

Let the force exerted by the unit of mass, at the distance k, be denoted by 1; the force exerted by the sun upon a body placed at the distance k will then be expressed by the mass M of the sun, or, in other words, by the number of units which its mass contains: but the mass of a planet attracted by the sun being denoted by m, this planet will exert an attraction upon the sun, which will, for a similar reason, be expressed by its mass m: moreover, since the two forces M and m tend to cause the approach of the two bodies, their effect upon the relative motion of the bodies will be the same as if the force $\mathrm{M}+m$ were concentrated in the sun, and acted on the planet at the distance k. When this distance varies and becomes equal to r, the intensity of the force will likewise vary. Let R denote its intensity at the distance r; the assumed hypothesis will give the proportion

$$\mathrm{M}+m : \mathrm{R} :: \frac{1}{k^2} : \frac{1}{r^2};$$

whence,

$$\mathrm{R}=\frac{k^2(\mathrm{M}+m)}{r^2} \;\ldots\ldots\; (265).$$

Such is the value of the attractive force which, acting at the distance r, will cause the bodies to approach each other.

505. The value thus determined corresponds to that of the incessant force which we have hitherto represented by R: we therefore have

$$\int \mathrm{R}dr=\int \frac{k^2(\mathrm{M}+m)}{r^2}dr;$$

putting, for brevity,

$$k^2(\mathrm{M}+m)=\mathrm{M}' \;\ldots\ldots\; (266),$$

the preceding equation will be reduced to

$$\int \mathrm{R}dr=\int \frac{\mathrm{M}'dr}{r^2} \;\ldots\ldots\; (267);$$

but since the quantities M and m and the distance k remain invariable, the quantity M′ will be constant; the equation (267) may therefore be readily integrated, and will give

$$\int \mathrm{R}dr = \frac{-\mathrm{M}'}{r} + c:$$

replacing $\int \mathrm{R}dr$ by this value, and $b-2c$ by b', the equations (259) and (260) will become

$$\frac{r^2 d\varphi^2 + dr^2}{dt^2} = b' + \frac{2\mathrm{M}'}{r} \ldots\ldots (268),$$

$$d\varphi = \frac{adr}{r\sqrt{(b'r^2 - a^2 + 2\mathrm{M}'r)}} \ldots\ldots (269).$$

506. To determine the value of the constant b', or its equal $b-2c$, we observe that the equations (258) and (268) give, by comparison,

$$\frac{dx^2 + dy^2}{dt^2} = \frac{r^2 d\varphi^2 + dr^2}{dt^2} = b' + \frac{2\mathrm{M}'}{r};$$

and since $\sqrt{(dx^2 + dy^2)}$ is equal to ds, the element of the curve, it appears that the quantity $\frac{r^2 d\varphi^2 + dr^2}{dt^2}$ is equal to $\left(\frac{ds}{dt}\right)^2$, or equal to the square of the velocity estimated in the direction of the tangent to the curve; thus, denoting this velocity by v, the equation (268) will become

$$v^2 = b' + \frac{2\mathrm{M}'}{r} \ldots\ldots (270).$$

If V represent the velocity at a given instant, and λ the corresponding value of the radius vector, the equation (270) will contain but a single unknown quantity b', whose value will result

$$b' = \mathrm{V}^2 - \frac{2\mathrm{M}'}{\lambda}.$$

507. The constant a may also be determined in functions of the initial velocity; for, by replacing $\frac{d\varphi}{dt}$ in the formula

$$v^2 = \frac{r^2 d\varphi^2 + dr^2}{dt^2},$$

by its value $\frac{a}{r^2}$ deduced from equation (264), we shall obtain

$$v^2 = \frac{dr^2}{dt^2} + \frac{a^2}{r^2} \ldots\ldots (271).$$

The identity of these expressions for the ranges corresponding to the angles α and α', evidently proves that the ranges will be equal when the two angles of projection are complements of each other.

525. To determine the angle of projection which corresponds to the greatest range, we remark that the range is in general expressed by $2h \sin 2\alpha$, and that this expression will become a maximum when the angle 2α is equal to 90°; hence it follows that a projectile in vacuo will have the greatest range upon a horizontal plane when the angle of projection is equal to 45°.

The supposition of $2\alpha = 90°$ gives $\sin 2\alpha = 1$; consequently, the expression for the range then becomes equal to $2h$; or the range corresponding to the angle of 45° is equal to twice the height due to the velocity of projection.

Let this range be denoted by P; we shall have

$$h = \tfrac{1}{2}\text{P} \; \ldots\ldots \; (292).$$

To determine the value of the coefficient h, the projectile may be thrown in a direction forming an angle of 45° with the horizontal plane, and the corresponding range may then be measured. If this range be represented by P, the value of h will immediately result from equation (292). In fire-arms, the coefficient h serves as a measure of the force of the powder, since the extent of the range evidently depends on the intensity of the force of projection.

526. The quantity h having been determined by taking the mean result of a large number of experiments, we substitute its value in equation (289), which will thus become

$$y = x \text{ tang } \alpha - \frac{x^2}{2\text{P} \cos^2 \alpha}.$$

If we represent by P′ the range corresponding to an angle α', the equation (291) will give

$$\text{P}' = 2h \sin 2\alpha' \; \ldots\ldots \; (293);$$

or, replacing h by its value $\tfrac{1}{2}$P (292), we find

$$\text{P}' = \text{P} \sin 2\alpha'.$$

This relation will determine the range P′ corresponding to the angle α', when the value of the maximum range has been previously ascertained; and, in general, we can calculate the range P' which corresponds to an angle α', from a knowledge of the range P'' given by any other angle α''; for, since

$$P'=P\sin 2\alpha',\quad P''=P\sin 2\alpha'',$$

we obtain, by division,

$$\frac{P'}{P''}=\frac{\sin 2\alpha'}{\sin 2\alpha''};$$

if, therefore, the range P'' corresponding to the angle α'' be determined by measurement, the value of P' corresponding to α' will result immediately from the preceding equation.

527. The value of h (292), being substituted in equation (288), will give, for the value of the initial velocity,

$$V=\surd(Pg)=\surd(32\tfrac{1}{6}\text{ ft.}\times P).$$

If, for example, the range corresponding to an angle of 45° were equal to 1000 feet, we should find

$$V=\surd(1000\text{ ft.}\times 32\tfrac{1}{6}\text{ ft.})=179.3\text{ ft., nearly.}$$

528. If, on the contrary, the initial velocity and angle of projection were given, we might determine the range: for example, let the initial velocity be supposed equal to 200 feet per second, and the angle of projection 15°; we first determine h from the following formula, deduced from (288),

$$h=\frac{V^2}{2g}=\frac{(200\text{ ft.})^2}{64\tfrac{1}{3}\text{ ft.}}=621.7\text{ ft.};$$

and the range P' will then become, (293),

$$P'=2\times 621.7\text{ ft.}\times\sin 30^\circ=261.7\text{ ft.}$$

529. The problem may also be presented under the following form:—Having given the initial velocity and the co-ordinates $x'=AB$, and $y'=BC$, of a point C (*Fig.* 193), it is required to determine the angle of projection such that the trajectory may pass through a given point C. The equation $V=\surd(2gh)$ will determine the value of h; and since the co-ordinates x' and y' should satisfy the equation (287), we shall have by substituting x' and y' for x and y,

$$y'=x'\operatorname{tang}\alpha-\frac{x'^2}{4h\,.\cos^2\alpha}\;\ldots\ldots\;(294).$$

In this equation the quantity α is alone undetermined: making $\operatorname{tang}\alpha=z$, we have

$$\cos\alpha=\frac{1}{\sec\alpha}=\frac{1}{\sqrt{(1+\text{tang}^2\alpha)}}=\frac{1}{\sqrt{(1+z^2)}};$$

and by substituting these values in equation (294) we find

$$y'=x'.z-\frac{x'^2}{4h}(1+z^2)\ \ldots\ldots\ (295).$$

This equation being resolved with reference to z, will give two values which determine the two angles of projection corresponding to the directions in which the projectile should be thrown in order that it may strike the point C; we select the greater of these two angles when we wish to crush the object upon which the projectile falls, as the vertical velocity at the point C will then be the greatest.

It may occur, that instead of the line CB, we have given the angle CAB subtended by the object CB. Let this angle be denoted by φ; we shall have

$$\text{CB}=x'\,\text{tang}\,\varphi=y';$$

this value of y', being introduced into equation (295), transforms it into

$$\text{tang}\,\varphi=z-\frac{x'}{4h}(1+z^2);$$

from which we deduce

$$z=\frac{2h}{x'}\pm\sqrt{\left(\frac{4h^2}{x'^2}-\frac{4h\,\text{tang}\,\varphi}{x'}-1\right)}.$$

Of the Motions of Projectiles in a Resisting Medium.

530. The theory of projectiles in vacuo, which has been examined in the preceding paragraphs, affords results which differ greatly from those obtained by direct experiments performed in the atmosphere: these discrepances are very considerable when the velocity of projection is great, and are to be attributed to the resistance opposed by the atmosphere to the motion of a body. If this resistance, represented by R, be supposed, as in Art. 412, to vary in the duplicate ratio of the velocity, we shall have

$$\text{R}=mv^2.$$

The resistance R at each point of the trajectory will be exerted in the direction of the element of the curve, but in an opposite direction to that of the motion; and the force R will form with the axes of co-ordinates the same angles as the element ds. Thus, denoting by α, β, and γ the angles included between the tangent to the curve at any point and the co-ordinate axes, the components of R will be expressed by

$$R\cos\alpha, \quad R\cos\beta, \quad R\cos\gamma.$$

To obtain expressions for these cosines, let mm' (*Fig.* 194) represent an element ds of the curve: the projection of this element on the axis of z will be equal to $m'n$. But the triangle $m'mn$ gives the proportion

$$1 : \cos mm'n :: mm' : m'n;$$

or,

$$1 : \cos\gamma :: ds : dz;$$

hence,

$$\cos\gamma = \frac{dz}{ds};$$

and the component of R in the direction of the axis of z, will therefore be expressed by

$$R\frac{dz}{ds}.$$

We attribute the negative sign to this component, because the tendency of the force R, while the projectile is moving from m to m', will be to diminish the co-ordinate z. For a similar reason the other components of the resistance R should be affected with the negative sign.

531. An analogous course of reasoning will give

$-R\frac{dx}{ds}$ for the component of R in the direction of the axis of x,

$-R\frac{dy}{ds}$ for the component in the direction of y.

Thus, the equations expressing the circumstances of the motion will be

$$\frac{d^2x}{dt^2} = -R\frac{dx}{ds},$$

$$\frac{d^2y}{dt^2} = -R\frac{dy}{ds},$$

$$\frac{d^2z}{dt^2} = -R\frac{dz}{ds} - g.$$

From the first two we obtain, by division,

$$\frac{d^2y}{d^2x}=\frac{dy}{dx};$$

or,

$$\frac{d^2y}{dy}=\frac{d^2x}{dx} \ldots\ldots (296);$$

and by integration,

$$\log dy=\log dx+\log a=\log adx.$$

Passing from logarithms to numbers, we find

$$dy=adx;$$

and by a second integration,

$$y=ax+b;$$

hence we conclude, that the projection of the trajectory on the plane of x, y is a right line, and therefore that the trajectory is contained in a vertical plane.

532. If we resume the consideration of the problem with this restriction, that the curve shall be confined to a vertical plane, it will only be necessary to employ the two equations

$$\frac{d^2x}{dt^2}=-R\frac{dx}{ds}, \quad \frac{d^2y}{dt^2}=-R\frac{dy}{ds}-g.$$

It has already been remarked, that the vertical component of the resistance $R\frac{dy}{ds}$ should be affected with the negative sign, since this resistance tends to diminish the co-ordinate; but this tendency will only exist whilst the projectile is describing the ascending branch of the trajectory. If, on the contrary, the projectile be supposed at a point M″ in the descending branch (*Fig.* 194), the resistance, being exerted in the direction from M″ to M′, would tend to increase the co-ordinate y. It might, therefore, appear that the component $R\frac{dy}{ds}$ should change its sign; but since dy becomes negative in the second branch of the curve, the vertical component will still be expressed by $-R\frac{dy}{ds}$,

If the quantity R in the preceding equations be replaced by its value mv^2, they will become

$$\frac{d^2x}{dt^2}=-mv^2\frac{dx}{ds},\quad \frac{d^2y}{dt^2}=-mv^2\frac{dy}{ds}-g.$$

The quantity v^2 may be eliminated by means of the equation

$$v^2=\frac{ds^2}{dt^2};$$

and we shall have

$$\frac{d^2x}{dt^2}=-m\frac{ds^2}{dt^2}\times\frac{dx}{ds}\;.....\;(297),$$

$$\frac{d^2y}{dt^2}=-m\frac{ds^2}{dt^2}\times\frac{dy}{ds}-g\;.....\;(298).$$

533. The first of these equations being multiplied by dt, gives

$$\frac{d^2x}{dt}=-mds\;.\;\frac{dx}{ds}\;.\;\frac{ds}{dt},$$

or,

$$\frac{d^2x}{dt}=-mds\frac{dx}{dt};$$

from this equation, we deduce

$$\frac{\frac{d^2x}{dt}}{\frac{dx}{dt}}=-mds;$$

and by integration,

$$\log\frac{dx}{dt}=-ms+C.$$

534. Let A represent the number whose logarithm is equal to C, and e the base of the Naperian system; we shall have

$$C=\log A,\quad \log e=1;$$

the preceding equation may therefore be transformed into

$$\log\frac{dx}{dt}=-ms\log e+\log A,$$

or,

$$\log\frac{dx}{dt}=\log e^{-ms}+\log A=\log Ae^{-ms};$$

passing from logarithms to numbers, we have

$$\frac{dx}{dt}=Ae^{-ms}\;.....\;(299).$$

535. To determine the constant A, let V represent the initial velocity, and α the angle formed by the direction of the initial impulse with the axis of x. The component of V in the direction of this axis will be expressed by V cos α. But when $s=0$, $\frac{dx}{dt}$ will express the component of the initial velocity along the axis of x: hence the preceding equation will, on this supposition, be reduced to

$$V \cos \alpha = Ae^{0} = A.$$

This value substituted in (299) converts it into

$$\frac{dx}{dt} = V \cos \alpha \,.\, e^{-ms} \;\ldots\ldots\; (300).$$

536. Since this equation contains three variables, we must obtain a second relation between them, in order to render the integration possible. For this purpose, the equations (297) and (298) may be written under the form

$$\frac{dx}{ds} = \frac{-\frac{d^2x}{dt^2}}{m\frac{ds^2}{dt^2}}, \quad \frac{dy}{ds} = \frac{-\left(\frac{d^2y}{dt^2}+g\right)}{m\frac{ds^2}{dt^2}};$$

the quantity ds may be eliminated immediately by division; and we thus obtain

$$\frac{dy}{dx} = \frac{\frac{d^2y}{dt^2}+g}{\frac{d^2x}{dt^2}}.$$

From this equation we deduce

$$g = \frac{dy}{dx} \cdot \frac{d^2x}{dt^2} - \frac{d^2y}{dt^2};$$

or, by reduction,

$$gdt^2 = \frac{dyd^2x - dxd^2y}{dx} \;\ldots\ldots\; (301).$$

The second member being divided by $-dx$ becomes the exact differential of $\frac{dy}{dx}$; and the equation (301) may therefore be written

$$gdt^2 = -dx \,.\, d\left(\frac{dy}{dx}\right).$$

If, for greater simplicity, we make $\frac{dy}{dx}=p$, there will result

$$gdt^2=-dx\,.\,dp \;.\;.\;.\;.\;.\; (302);$$

and eliminating dt, by means of equation (300), we find

$$g=-V^2\cos^2\alpha\,.\,e^{-2ms}\,.\,\frac{dp}{dx} \;.\;.\;.\;.\;.\; (303).$$

537. This equation still contains three variables; but one of them may be readily eliminated by means of the relation $ds=\sqrt{(dx^2+dy^2)}$; in which, replacing dy by its value pdx, we obtain

$$ds=dx\sqrt{(1+p^2)} \;.\;.\;.\;.\;.\; (304);$$

and consequently, by eliminating dx between this equation and (303), there will result

$$dp\sqrt{(1+p^2)}=\frac{-ge^{2ms}ds}{V^2\cos^2\alpha} \;.\;.\;.\;.\;.\; (305).$$

Integrating, we have

$$\tfrac{1}{2}p\sqrt{(1+p^2)}+\tfrac{1}{2}\log\left[p+\sqrt{(1+p^2)}\right]=C-\frac{ge^{2ms}}{2mV^2\cos^2\alpha}\ldots(306);$$

and by making $C=\frac{1}{2}B$, and suppressing the common divisor 2, we obtain

$$p\sqrt{(1+p^2)}+\log\left[p+\sqrt{(1+p^2)}\right]=B-\frac{ge^{2ms}}{mV^2\cos^2\alpha} \;.\;.\;.\;.\;.\; (307).$$

To determine the value of the constant B, we observe that $\frac{dy}{dx}$ expresses the trigonometrical tangent of the angle formed by the element of the curve with the axis of x. At the point A, the origin of the motion, this angle is denoted by α, the quantity t being at the same time equal to zero; we shall therefore have

$$x=0,\quad y=0,\quad s=0,\quad p=\text{tang}\,\alpha.$$

These values of s and p being substituted in the preceding equation give

$$B=\text{tang}\,\alpha\sqrt{(1+\text{tang}^2\alpha)}$$
$$+\log\left[\text{tang}\,\alpha+\sqrt{(1+\text{tang}^2\alpha)}\right]+\frac{g}{mV^2\cos^2\alpha}:$$

the value of the constant B in equation (307) may therefore be regarded as known.

538. If we eliminate e^{2ms} between the equations (303) and (307), we shall obtain

$$dx=\frac{dp}{m[p\sqrt{1+p^2}+\log(p+\sqrt{1+p^2})-B]}\;\ldots\ldots(308).$$

The two members of this equation being multiplied by the corresponding members of the equation

$$\frac{dy}{dx}=p,$$

there will result

$$dy=\frac{pdp}{m[p\sqrt{1+p^2}+\log(p+\sqrt{1+p^2})-B]}\;\ldots\ldots(309).$$

539. To determine the time t, we substitute in the equation

$$dt^2=-\frac{dp\,.\,dx}{g},$$

the value of dx, given by equation (308), and we thus obtain

$$dt^2=\frac{-dp^2}{mg[p\sqrt{1+p^2}+\log(p+\sqrt{1+p^2})-B]}\;\ldots\ldots(310);$$

or, by changing the signs of the numerator and denominator,

$$dt^2=\frac{dp^2}{mg[-p\sqrt{1+p^2}-\log(p+\sqrt{1+p^2})+B]}.$$

In extracting the square root of the two members of this equation, the second might be affected with the double sign, but in the present instance we shall attribute to it the negative sign. For, since every equation between two variables t and p may be regarded as that of a curve, of which t is the absciss, and p the ordinate, if p increases whilst t diminishes, the elements dt and dp will necessarily be affected with contrary signs. But, in the present case, it is obvious that whilst t augments, the quantity p, which expresses the trigonometrical tangent of the angle formed by the element of the curve with the axis of x, constantly diminishes in the ascending branch of the trajectory, which is the one at present under consideration; hence, we shall have

$$dt=-\frac{dp}{\sqrt{mg[-p\sqrt{1+p^2}-\log(p+\sqrt{1+p^2})+B]}}\;\ldots\ldots(311).$$

540. The expression for the velocity can now be obtained in functions of p; for, the velocity resulting from the equation

$$v=\frac{ds}{dt}=\frac{\sqrt{(dx^2+dy^2)}}{dt}=\frac{dx}{dt}\sqrt{(1+p^2)},$$

we obtain, after replacing dx and dt by their respective values,

$$v=\frac{\sqrt{\frac{g}{m}}\times\sqrt{(1+p^2)}}{\sqrt{-p\sqrt{(1+p^2)}-\log[p+\sqrt{(1+p^2)}]+\text{B}}}.$$

541. We can also express the arc s in functions of p; for the equation (307) gives

$$e^{2ms}=\frac{m\text{V}^2\cos^2\alpha}{g}\Big(\text{B}-p\sqrt{(1+p^2)}-\log[p+\sqrt{(1+p^2)}]\Big).$$

Taking the logarithms, and reducing, we obtain

$$s=\frac{\log\left(\frac{m\text{V}^2\cos^2\alpha}{g}[\text{B}-p\sqrt{1+p^2}-\log(p+\sqrt{1+p^2})]\right)}{2m}.$$

542. To obtain the equation of the trajectory, it would be necessary to integrate equations (308) and (309): these integrations cannot be effected except by the aid of series. Nevertheless, by employing equations (308) and (309), the curve may be constructed approximatively by points.

For this purpose, we will write those equations under the form,

$$dx=\varphi p\,.\,dp\,\ldots\ldots\,(312),$$
$$dy=\psi p\,.\,dp\,\ldots\ldots\,(313);$$

in which φp and ψp represent certain known functions of p.

The first of these equations gives

$$\frac{dx}{dp}=\varphi p:$$

the quantity $\frac{dx}{dp}$ represents the tangent of the angle included between the axis of abscisses and the element of a curve whose co-ordinates are denoted by p and x respectively. We will first construct this curve, which will serve to determine points in the trajectory. It is distinguished by the name of the *auxiliary curve.*

Having drawn two rectangular axes Ap and Ax (*Fig.* 195), lay off from A to B a distance $AB=\text{tang}\,\alpha$; the point B will appertain to the auxiliary curve, since the ordinate $x=0$ corresponds to the absciss $p=\text{tang}\,\alpha$.

If the line AB be divided into equal parts BB′, B′B″, &c., each of these parts being represented by dp, it will be easy to construct approximatively the points M, M′, M″, &c. of the auxiliary curve, corresponding to the points B, B′, B″, &c. For, if we suppose the points B, B′, B″, &c. to be exceedingly near to each other, we may regard the arcs M′B, M″M′, M‴M″, &c. of the curve as coinciding with the tangents drawn to the points M′, M″, M‴, &c. The ordinates M′B′, M″B″, M‴B‴, &c. may then be calculated; for, the trigonometrical tangent of the angle formed by the element of the curve with the axis of p, being represented in general by $\frac{dx}{dp}$, its value will always be given by means of equation (312), whenever we assume a value for p. Thus, if we wish to determine the trigonometrical tangent of the angle $M'Bp$ included between the tangent at M′, and the axis of abscisses, since the absciss of the point M′ is $AB'=AB-BB'=\text{tang}\,\alpha-dp$, it will be necessary to change p into $\text{tang}\,\alpha-dp$, in the value $\varphi p=\frac{dx}{dp}$, given by equation (312): we thus obtain

$$\text{tang}\,M'Bp=\varphi(\text{tang}\,\alpha-dp)\,;$$

whence,

$$\text{tang}\,M'BB'=-\varphi(\text{tang}\,\alpha-dp).$$

The ordinate M′B′ being expressed by $BB'\times\text{tang}\,M'BB'$, we shall have

$$M'B'=BB'\times\text{tang}\,M'BB'\,;$$

or,

$$M'B'=dp\times-\varphi(\text{tang}\,\alpha-dp).$$

Thus, the point M′, of the auxiliary curve BC, may be constructed by means of the co-ordinates

$$AB'=\text{tang}\,\alpha-dp,$$

and

$$B'M'=dp\times-\varphi(\text{tang}\,\alpha-dp).$$

To determine a third point M″, we make $AB'' = \text{tang}\,\alpha - 2dp$; and by the same course of reasoning prove that the trigonometrical tangent of the angle M″M′O is expressed by $-\varphi(\text{tang}\,\alpha - 2dp)$, and consequently,

$$M''O = dp \times -\varphi(\text{tang}\,\alpha - 2dp)$$

substituting this value and that of M′B′ in the equation

$$M''B'' = M'B' + M''O,$$

given by an inspection of the figure, we find

$$M''B'' = -dp\,.\,\varphi(\text{tang}\,\alpha - dp) - dp\,.\,\varphi(\text{tang}\,\alpha - 2dp).$$

To calculate the ordinate M‴B‴ which corresponds to the absciss $AB''' = \text{tang}\,\alpha - 3dp$, it will only be necessary to add to the value of M″B″ that of the portion M‴O′, which, by an investigation similar to the preceding, may be proved equal to $-dp\,.\,\varphi(\text{tang}\,\alpha - 3dp)$: thus we have

$$B'''M''' = -dp\,.\,\varphi(\text{tang}\,\alpha - dp) - dp\,.\,\varphi(\text{tang}\,\alpha - 2dp) \\ - dp\,.\,\varphi(\text{tang}\,\alpha - 3dp).$$

In this manner we may determine a series of points which will appertain to the auxiliary curve, the co-ordinates of which are x and p. Connecting these points by right lines, we form a polygon BM′M″M‴, &c., which will coincide more nearly with the curve, in proportion as dp has a smaller value assigned to it.

By performing similar operations with reference to the equation

$$dy = \psi p\,.\,dp,$$

we may construct a second auxiliary curve BD, the co-ordinates of which will represent the quantities p and y. The co-ordinates mb and lb, which in these two curves correspond to the same value of p, will represent the two co-ordinates of a point in the trajectory; so that by taking the co-ordinates B′M′, B″M″, B‴M‴, &c. of the first curve as the abscisses of the trajectory, its ordinates will be represented by the lines B′L′, B″L″, B‴L‴, &c.

Of the different Methods of measuring the Effects of Forces.

543. It has been remarked (Art. 388), that two forces F and F' applied to the same body are proportional to the velocities which they can impress upon that body. Let it now be supposed that these forces are applied to different masses.

If two equal forces acting in opposite directions be applied to equal and spherical masses M and M', they will communicate to these masses the equal velocities V and V'; and if these masses be supposed to impinge directly upon each other, they will mutually destroy each other's motion, and an equilibrium will ensue, since the circumstances of motion in each are precisely similar. But if the mass M be supposed equal to nM', and V' greater than V, we may regard M as composed of n masses $m', m'', m''', \ldots\ldots m^{(n)}$, each equal to the mass M'. In consequence of the mutual connexion of the different parts of the system, each of the masses m', m'', m''', &c. must move with the same velocity V, so that if the body M be supposed to pass over the space of three feet in one second of time, each of the masses m', m'', m''', &c. will likewise pass over a distance of three feet in one second; or, if V represent the velocity of the mass M, V will likewise express the velocity of each of the masses m', m'', m''', &c. But if the mass m', moving with the velocity V, should impinge against the equal mass M', which moves with the velocity V', it would destroy a portion of the velocity of the second body equal to V; and if, at the same instant, the mass m'', acting by its connexion with the other masses, should impinge against the body M', it would likewise destroy a portion of the velocity V', equal to V: and the same may be said of the other masses m''', m^{iv}, &c. Thus, the joint effect of the several masses $m', m'', m''', \ldots\ldots m^{(n)}$, would be to destroy in the mass M' a velocity represented by nV. If we suppose the velocity V' to be entirely destroyed, an equilibrium will ensue, and it will be necessary that $V'=nV$.

By eliminating n between this equation and the relation $M=nM'$, we obtain the proportion

$$M : M' :: V' : V;$$

from which we conclude *that an equilibrium will ensue when two bodies are caused to impinge directly against each other, with velocities inversely proportional to their masses.*

544. It may be readily demonstrated that the same proposition is equally true when the mass M does not contain the mass M′ an exact number of times. For, if the mass M be supposed to contain n masses, each of which is equal to m, and the mass M′ to contain a number of these equal masses, denoted by n'; each mass m contained in M, will destroy a portion V of the velocity V′ of a mass m contained in M′; or, since M′ is supposed to contain n' masses, each of which is equal to m, the mass m, moving with the velocity V, will destroy in $M'=n'm$ a velocity expressed by $\frac{V}{n'}$; and since the other equal masses contained in the body M will produce similar effects, the entire velocity destroyed in M′ by M will be equal to $\frac{V}{n'}$ repeated as many times as the mass m is contained in M, or it will be equal to $\frac{V}{n'}\times n$: if we suppose the velocity V′ to be entirely destroyed, we must have

$$V'=V\frac{n}{n'};$$

or,

$$V : V' :: n' : n :: mn' : mn;$$

and replacing mn, mn', by their values M, M′, we obtain the proportion

$$V : V' :: M' : M:$$

whence the truth of the proposition is manifest.

545. Since the masses of the bodies are in the inverse ratio of their velocities when an equilibrium is produced, it follows, that if the bodies have equal volumes, and unequal densities, their velocities will be in the inverse ratio of their densities.

546. Let F represent a force which impresses a velocity V upon a mass M: if the same force be supposed to act upon a mass M times less, and which will consequently be represented by $\frac{M}{M}=1$, this force will communicate to the mass

unity, a velocity M times greater than that communicated to the mass M: this velocity will therefore be expressed by MV. For a similar reason, the force F', which communicates to the mass M' a velocity V', would communicate to the mass $\frac{M'}{M'}=1$, a velocity represented by M'V'.

The velocities represented by MV and M'V' being communicated by the forces F and F' to the mass unity, it follows, from the principles enunciated in Art. 388, that we shall have the proportion

$$F : F' :: MV : M'V'.$$

The expressions MV and M'V' are called the *quantities of motion* communicated by the forces F and F'; and it should be recollected that the characters M, V, F, M', V', and F' represent abstract numbers, which merely express the number of times which the quantity under consideration contains the unit of its own species.

547. The unit of force being arbitrary, we may represent it by the quantity of motion which it produces. Thus, by supposing F' to represent this unit, we can replace F' by M'V' in the preceding proportion; and we thence infer that

$$F=MV.$$

548. When the force φ acts incessantly, it has been shown, Art. 388, that this force will be represented by the velocity which it would communicate in a unit of time, if the value of the force should become constant; hence we obtain, by substituting for V its value φ,

$$F=M\varphi.$$

If the mass M be supposed equal to unity, we shall have

$$F=\varphi;$$

consequently, ϕ represents the force exerted upon the unit of mass; the quantity ϕ is usually called the *accelerating* force, and F is called the *moving* force. When F is given, the value of φ can be determined by simply dividing by M, the mass moved.

549. It has been shown, Art. 163, that if g represent the force of gravity, P the weight of the body, and M its mass, we shall have

$$P=Mg;$$

eliminating M between this equation and the preceding, there results

$$F=P\frac{\phi}{g};$$

and if the incessant force ϕ be that of gravity, we have $\phi=g$; hence,

$$F=P;$$

and in this case the moving force is measured by the weight of the body upon which the force is exerted.

550. The writers upon Mechanics were long divided in opinion as to the proper measure of forces. This disagreement, like many others, arose entirely from a misapprehension of the signification of words.

The nature of forces being known to us only by the effects which they produce, we may with propriety measure these effects in different ways, according to the object which it is desired to accomplish. If, for example, it be proposed to determine the load which a man can support for an instant of time, it is evident that the force exerted by the man will be proportional to the weight which he can sustain, and may therefore be measured by this weight: but if we wish to measure the force of this man by the work which he can perform in a given time, we must adopt a measure for the force entirely different from the preceding: for, it might happen that a man absolutely weaker, but endued with a greater capacity of sustaining a continued effort, would give by his labour a result greater than that given by the first man, and might therefore be considered as actually possessed of greater force.

In this second method of considering the effects of forces, we regard them as proportional to the weight raised, and the height to which it is elevated in a given time; it being always understood that the effort necessary to overcome the weight is not supposed to vary with the elevation.

If, for example, two men raise the same weight, in the same time, to the heights of 600 and 200 yards respectively, we would, according to this method of estimating the effects of forces, regard the first as possessed of three times the force of the second.

Again, if, in the working day, one man can raise a weight of 50 lbs. through a height of 200 yards, and a second a weight of 25lbs. through a height of 400 yards, we should regard the two men, according to the present hypothesis, as possessed of equal strength, although the absolute strengths of the two might be very different; the strengths of the two individuals are here considered only with reference to the work done.

This method of estimating forces was adopted by Descartes. The difference in the opinions entertained by him and other geometers rested entirely on the definition of the word *force*. He contended that a force should be measured by the product of the mass into the square of the velocity. This consequence may be deduced from the definition of the effect of a force, adopted by Descartes, in the following manner.

Let P represent a weight, and h the height to which it can be raised in a given time: the force employed to raise it, according to the definition of Descartes, will be measured by the product

$$P\times h.$$

We can replace P in this expression by its value Mg (Art. 163), and we shall have

$$Ph=Mgh;$$

or, multiplying by 2,

$$2Ph=M\times 2gh;$$

and since the velocity v due to the height h is expressed by $\sqrt{(2gh)}$ (Art. 401), the preceding expression becomes

$$2Ph=Mv^2.$$

Having given a definition of the word force different from that adopted by Descartes, we shall not say that the force is measured by the product Mv^2, but that it is measured by the quantity of motion Mv which it is capable of producing, as has been explained in Art. 547; and to avoid confusion, we shall, according to ordinary usage, apply the term *living force* to the product Mv^2, of the mass by the square of the velocity.

551. The consideration of living forces is of great utility in estimating the effects produced by a machine. Thus, if

it were required to calculate the effect of a given fall of water, the force necessary to move a carriage on a given piece of ground, or the effort requisite to raise a given mass of coals from the bottom of a mine, we might in each case compare the effect of the moving force to the product of a certain weight by a given height, or to an expression of the form Ph, the double of which, as has been before shown, is equivalent to the product Mv^2.

Of the Direct Impact of Bodies.

552. Bodies are usually distinguished as elastic or unelastic. An elastic body is that which, when compressed by the application of an impulse, will resume its original figure with a force equal to that of compression, in virtue of a quality possessed by the body. An unelastic body, on the contrary, is one whose figure either undergoes no change by the action of a force applied to it, or which, if compressed, has no tendency to restore itself to its original form.

All natural bodies are found to partake more or less of these two qualities; there being none which are perfectly elastic, or perfectly unelastic.

Of the Direct Impact of Unelastic Bodies.

553. Let M and M′ (*Fig.* 196) represent two spherical unelastic bodies, which move in the direction from A to C. If the velocity of M be supposed to exceed that of M′, the former will overtake the latter, and will communicate to it a portion of its motion, until the velocities of the two bodies become equal. Let F and F′ represent the forces which communicate to the bodies M and M′ their respective velocities V and V′; since these forces can be represented by the quantities of motion which they produce (Art. 547), we shall have

$$F=MV, \quad F'=M'V';$$

and by compounding these two forces, their resultant will be expressed by

$$F+F'=MV+M'V'.$$

To obtain a second expression for $F+F'$, let v represent the common velocity of the two bodies after impact: we may regard the mass $M+M'$ as a single body, to which the velocity v has been imparted by the exertion of a force $F+F'$. We shall then have

$$F+F'=(M+M')v.$$

By equating these two values of $F+F'$, we obtain

$$(M+M')v=MV+M'V';$$

whence, we deduce

$$v=\frac{MV+M'V'}{M+M'}.$$

554. If the bodies move in opposite directions, we regard one of the velocities V' as negative, and we then have

$$v=\frac{MV-M'V'}{M+M'}.$$

The body M' being supposed at rest, and impinged against by the body M, V' will become equal to zero, and the preceding formula will reduce to

$$v=\frac{MV}{M+M'}.$$

If the bodies have equal masses and move in the same direction, we shall have $M=M'$; and consequently,

$$v=\tfrac{1}{2}(V+V'),$$

or, if they move in contrary directions,

$$v=\tfrac{1}{2}(V-V'):$$

and when the body M impinges upon an equal mass M' at rest, this expression reduces to

$$v=\tfrac{1}{2}V$$

Of the Direct Impact of Elastic Bodies.

555. We will first consider the circumstances of motion when an elastic spherical body impinges upon an immoveable plane AB (*Fig.* 197) in a direction perpendicular to the surface of the plane. At the instant when the body comes in contact with the plane, it will begin to experience a compression in the direction of the diameter ED, the point D being caused to approach the centre of the sphere. This

effect will continue until the velocity of the sphere is entirely destroyed; then, in virtue of the elasticity possessed by the body, an equal velocity will be generated in an opposite direction, the body at the same time resuming its original figure. Hence, the body will recoil with a velocity precisely equal to that with which it impinged upon the plane.

556. Let us next consider the impact of two elastic bodies M and M′ (*Fig.* 196), which move in the same direction from A towards C, with velocities represented by V and V′. That an impact may be possible, it is necessary that the velocity of M should exceed that of M′. When the body M overtakes M′, a mutual compression will commence, and will continue until the bodies have acquired a common velocity; so that a material point D of the body M (*Fig.* 198), which, in virtue of the velocity V, would have described the line DE, being retarded in its motion by the effect of the compression, will, instead of having reached the point E at the instant of maximum of compression, have only arrived at a point F; then the force of restitution, beginning to act upon the material point, will communicate to it a velocity in a direction opposite to that of the motion, equal to that which it has lost by the compression, and which would transfer it to the extremity G of a line FG = EF, whilst the body is resuming its original figure.

The velocity of the body being common to all its points, (Art. 443), if we represent this velocity before impact by DE, it may be represented after impact by

$$DE - GE = DE - 2FE.$$

557. To express these conditions analytically, let u represent the velocity common to all the particles of the two bodies at the moment of maximum compression. At this instant, the bodies may be regarded as unelastic, and the velocity u will therefore be given by the formula

$$u = \frac{MV + M'V'}{M + M'} \ \ldots\ldots \ (314).$$

The velocity lost by the body M during the compression, being equal to the velocity V diminished by that which remains at the instant of greatest compression, it will be expressed by $V - u$. Such will be the velocity lost at the moment of greatest compression, but the force of elasticity, tending to restore the figures of the bodies, will cause the body M to sustain an additional loss of velocity, represented by $V-u$; thus, the total loss of velocity experienced by M will be expressed by $2(V-u)$. Let v denote the velocity of the mass M after the impact; we shall have

$$v=V-2(V-u);$$

or, by reduction,

$$v=2u-V \ldots\ldots (315);$$

The body M′, at the instant of greatest compression, may likewise be regarded as unelastic, and will then have gained a velocity expressed by $u-V'$: for the velocity gained is evidently equal to the velocity u which the body has at this instant, diminished by the original velocity V′. The force of restitution, being then exerted, will cause the body to gain the additional velocity $u-V'$; whence, the entire gain of velocity by M′ will be equal to $2(u-V')$, and the velocity of M′, after collision, will therefore be expressed by

$$V'+2(u-V')=2u-V'.$$

Representing this velocity by v', we have

$$v'=2u-V' \ldots\ldots (316).$$

By substituting in equations (315) and (316) the value of u given by (314), we find

$$v=\frac{2(MV+M'V')}{M+M'}-V, \quad v'=\frac{2(MV+M'V')}{M+M'}-V';$$

from which, by reduction, we obtain

$$v=\frac{V(M-M')+2M'V'}{M+M'}, \quad v'=\frac{V'(M'-M)+2MV}{M+M'} \ldots\ldots (317).$$

If $M=M'$, we shall have

$$v=V', \quad v'=V \ldots\ldots (318).$$

These equations indicate, that when the masses are equal, the impact will cause them to exchange velocities.

558. If the bodies move in opposite directions, the velocity V′ may be regarded as negative in the preceding formulas, which then become

$$v=\frac{V(M-M')-2M'V'}{M+M'}, \quad v'=\frac{V'(M-M')+2MV}{M+M'} \ldots\ldots (319).$$

559. The bodies being supposed equal in mass, and moving in opposite directions, we make $M=M'$ in equations (319), which are thus reduced to

$$v=-V', \quad v'=V \ldots\ldots (320).$$

Hence we conclude that the bodies will recoil, having exchanged velocities.

560. When the bodies impinge in opposite directions, with equal velocities, the masses of the two being unequal, we make $V'=V$ in equations (319), and thus obtain

$$v=\frac{V(M-3M')}{M+M'}, \quad v'=\frac{V(3M-M')}{M+M}.$$

In this case, the motion of M will be entirely destroyed by the impact, if its mass be supposed triple that of M'; for when $M=3M'$, the first equation reduces to $v=0$: the same supposition gives $v'=2V$.

561. Lastly, the body M' being supposed at rest, and impinged against by an equal body M, we make $M=M'$, and $V'=0$, in equations (317), and we thus have

$$v=0, \quad v'=V:$$

hence, the body M will be brought to rest, and M' will acquire its entire velocity.

Of the Preservation of the Motion of the Centre of Gravity in the Impact of Bodies.

562. Let the two bodies M and M' be supposed to have arrived at the positions B and C (*Fig.* 199), immediately before impinging upon each other; and let S and S' represent their distances from the point A, and X the distance of their common centre of gravity from the same point. From the known property of the centre of gravity, we shall have

$$(M+M')X=MS+M'S';$$

and since the distances X, S, and S' vary with the time t, we shall obtain, by differentiating with reference to t,

$$(M+M')\frac{dX}{dt}=M\frac{dS}{dt}+M'\frac{dS'}{dt}.$$

ficients $\frac{dS}{dt}$ and $\frac{dS'}{dt}$ represent the velo-
s M and M′ at the instant when they have
ints B and C, the distances of which from the
epresented by S and S′ respectively. Let these
e denoted by V and V′, and that of the centre of
by $W=\frac{dX}{dt}$: we shall obtain, by substitution,

$$W=\frac{MV+M'V'}{M+M'} \quad \ldots\ldots (321).$$

Such is the expression for the velocity of the common centre of gravity before the impact: but immediately after the impact, the bodies, being found at the points B′ and C′, will have experienced a change in their velocities, and it is required to determine what effect has been produced upon the velocity of their centre of gravity. Let w denote the velocity of the common centre of gravity after impact, and x its distance from the point A, in the new positions of the bodies; the distances of the bodies from A being represented by s and s' respectively, and their velocities by U and U′, we shall have, as above,

$$(M+M')x=Ms+M's';$$

and by differentiating with reference to t, we find

$$(M+M')\frac{dx}{dt}=M\frac{ds}{dt}+M'\frac{ds'}{dt}.$$

Replacing $\frac{dx}{dt}$, $\frac{ds}{dt}$, and $\frac{ds'}{dt}$ by their respective values w, U, and U′, there results

$$w=\frac{MU+M'U'}{M+M'} \quad \ldots\ldots (322).$$

563. Two different cases may now be presented for examination; viz. the bodies may be elastic, or they may be unelastic; when they are unelastic, we have

$$U=u=U';$$

whence,

$$w=\frac{M+M'}{M+M'}u=u.$$

But it has been shown (Art. 553), that the velocity w common to the two bodies after the impact will be equal to

$$\frac{MV+M'V'}{M+M'};$$

this velocity being precisely equal to the velocity W, it follows that we shall have $w=W$; or, *the velocity of the common centre of gravity of two unelastic bodies is not affected by their impact.*

564. When the bodies are elastic, their velocities after impact will be expressed (Art. 557) by $2u-V$, and $2u-V'$.

Substituting these values of U and U′ in equation (322), we find

$$w=\frac{M(2u-V)+M'(2u-V')}{M+M'};$$

or, by reduction,

$$w=2u-\frac{MV+M'V'}{M+M'}:$$

replacing the second term of the second member by its value u, there will result

$$w=u;$$

or,

$$w=\frac{MV+M'V'}{M+M'}:$$

and eliminating the second member of this equation by means of equation (321), we find

$$w=W:$$

hence we conclude, *that in the impact of elastic bodies, as in that of unelastic bodies, the velocity of the centre of gravity is the same before and after impact.*

Of the Preservation of living Forces in the Impact of Elastic Bodies—Relative Velocity before and after Impact—Loss of living Force in the Collision of Unelastic Bodies.

565. The principle of the preservation of living forces in the collision of elastic bodies may be enunciated as follows: *When two elastic bodies impinge on each other, the sum of their living forces is the same before and after impact.*

Let V and V′ represent the velocities of the bodies before collision, and v and v' their velocities after collision; the sum of the living forces before the impact will be expressed by $MV^2+M'V'^2$; and it is required to prove that this sum is equal to $Mv^2+M'v'^2$, the sum of the living forces after the impact.

It has been shown (Art. 557), that the velocities v and v', after impact, are given by the equations

$$v=2u-V,\quad v'=2u-V';$$

hence,

$$Mv^2+M'v'^2=M(2u-V)^2+M'(2u-V')^2;$$

and by performing the operations indicated in the second member, we have

$$Mv^2+M'v'^2=MV^2+M'V'^2$$
$$+4(Mu^2+M'u^2-MVu-M'V'u) \,\ldots\ldots\, (323):$$

but the terms included within the brackets mutually destroy each other, in consequence of the relation (314),

$$u=\frac{MV+M'V'}{M+M'};$$

for, by clearing the denominator, and multiplying by u, we find

$$Mu^2+M'u^2=MVu+M'V'u;$$

consequently, the equation (323) will reduce to

$$Mv^2+M'v'^2=MV^2+M'V'^2.$$

This equation may be written under the form

$$Mv^2+M'v'^2-MV^2-M'V'^2=0;$$

from which we conclude that when elastic bodies impinge on each other, the difference between the sums of their living forces before and after impact, will be equal to zero.

566. The relative velocity of the two bodies is the velocity with which they approach towards, or recede from, each other; and another remarkable property of elastic bodies consists in the equality of their relative velocities before and after impact. This may be proved by subtracting the equations

$$v=2u-V,\quad v'=2u-V';$$

from which we obtain

$$v-v'=-(V-V');$$

hence v' exceeds v by the same quantity that V surpasses V′;

and the bodies will therefore separate after impact, with a velocity precisely equal to that with which they approached.

567. In the collision of unelastic bodies, the difference between the sums of the living forces before and after impact will not be equal to zero; but it will be equal to the sum of the living forces of the bodies when moving with the velocities lost or gained.

This theorem is due to Carnot, and may be demonstrated in the following manner:

The velocities lost and gained by M and M′ respectively, being equal to $V-u$, $u-V'$, if the masses were moved with these velocities, their living forces would be expressed by

$$M(V-u)^2, \quad M'(u-V')^2;$$

performing the operations indicated, we shall have

$$M(V-u)^2+M'(u-V')^2=$$
$$MV^2+M'V'^2+(M+M')u^2-2u(MV+M'V') \ldots\ldots (324);$$

eliminating $MV+M'V'$, by means of the equation

$$u=\frac{MV+M'V'}{M+M'},$$

the second member of equation (324) will reduce to

$$MV^2+M'V'^2-(M+M')u^2,$$

and we shall therefore have

$$M(V-u)^2+M'(u-V')^2=MV^2+M'V'^2-(M+M')u^2;$$

hence the truth of the theorem enunciated becomes apparent.

Principle of D'Alembert.

568. When the several bodies which compose a system are connected together in any manner, and subjected to the action of different forces, this connexion will in general prevent each body from taking the motion which would have been communicated to it if the connexion had not existed. For example, if several material points M, M′, M″, &c. (*Fig.* 200) be attached to an inflexible right line AL, moveable about the point A, it is evident that these points, being unable to move except with the line AL, will, when acted on by the force of gravity, oscillate together about the point A, describing arcs

proportional to their distances from A, and will at the end of a certain time be brought into the positions K, K′, K″, &c.; whereas, if the points were unconnected, being merely attached to the point A, they would, from the principles of the simple pendulum, explained in Art. 471, oscillate in very unequal times, depending on their distances from the point A. Moreover, if we resolve each of the several forces which are exerted in vertical directions upon the points M, M′, M″, &c., into two components, one of which shall act along the line AL, and the other in a direction perpendicular to this line; the latter component will alone tend to communicate motion to the point; and since the several perpendicular components, exerted on the different points, will be equal to each other, they would communicate in the instant of time dt equal velocities to the points M, M′, M″ &c., if these points were unconnected. But in consequence of their connexion, the velocities assumed are evidently proportional to their distances from the point A.

569. It thus appears, that the effective velocities assumed by the several parts of the system differ from the velocities impressed, and hence the circumstances of the motion can only be discovered when we have succeeded in expressing the effective velocities in functions of the velocities impressed. This object is readily accomplished with the assistance of a dynamical principle first employed by D'Alembert.

570. Let v, v', v'', &c. represent the velocities which would be impressed by certain forces on the bodies M, M′, M″, &c., if they were perfectly free, and u, u', u'', &c. the velocities assumed by these bodies in consequence of their connexion. The velocity v being resolved into two components, one of these components may be assumed arbitrarily, and the second will then become determinate. Let the effective velocity u be assumed as the arbitrary component of the impressed velocity v, and denote the other component by U. Making a similar decomposition of the other velocities v', v'', &c., we have

u and U for the components of v,
u' and U′ for those of v',
u'' and U″ for those of v'',
&c. &c. &c.;

and the quantities of motion impressed upon the system, which are Mv, $M'v'$, $M''v''$, &c., will become, after the decomposition,

$$Mu,\ M'u',\ M''u'',\ \&c.,$$
$$MU,\ M'U',\ M''U'',\ \&c.$$

But, in consequence of the connexion of the different parts of the system, these quantities of motion will be reduced to

$$Mu,\ M'u',\ M''u'',\ \&c.\ ;$$

hence, it is necessary that the quantities of motion MU, $M'U'$, $M''U''$, &c. should destroy each other, or should produce an equilibrium. For, if it were otherwise, we might combine the resultant of the quantities of motion MU, $M'U'$, $M''U''$, &c. with the quantities of motion Mu, $M'u'$, $M''u''$, &c.; thus the effective velocities of the several parts of the system would no longer be represented by u, u', u'', &c., which is contrary to the hypothesis.

571. It may be observed that the products MU, $M'U'$, $M''U''$, &c. express the quantities of motion due to the velocities lost or gained by the several bodies. For the velocity v may be replaced by its two components u and U; the former of which expresses the effective velocity of the body M, and the latter represents that velocity which, combined with u, would produce the impressed velocity. Thus, U is a velocity introduced or destroyed in the system by the connexion of its parts.

The general principle may therefore be enunciated in the following manner: *It is necessary that the quantities of motion due to the velocities lost or gained should be such as would maintain the system in equilibrio.*

572. It has been remarked that the quantity of motion Mv may be resolved into the two components Mu and MU; and since an equilibrium will always subsist between three forces, one of which is equal and directly opposed to the resultant of the other two, it follows that the forces represented by Mu and MU will sustain in equilibrio a force equal and opposite to Mv; and consequently, that the force Mv will sustain in equilibrio two forces which are respectively equal and opposite to Mu and MU.

The same remarks being applicable to the other forces, it appears that the forces Mv, $M'v'$, $M''v''$, &c. will sustain in equilibrio two systems of forces which are equal and directly opposed to the forces

$$Mu,\ M'u',\ M''u'',\ \&c.,$$
$$MU,\ M'U',\ M''U'',\ \&c.$$

But the forces MU, $M'U'$, $M''U''$, &c. destroy each other; and hence we obtain a second enunciation of the principle of D'Alembert, viz.; *An equilibrium will subsist between the quantities of motion* Mv, $M'v'$, $M''v''$, *&c. impressed upon the several bodies, and the effective quantities of motion* Mu, $M'u'$, $M''u''$, *&c., the latter being applied in directions contrary to those of the motions actually assumed.*

573. This principle is equally true, whether the velocities v, v', v'', &c. are finite velocities, acquired by the masses M, M′, M″, &c. during a finite time, or communicated instantaneously by forces of impulsion; or, when these velocities are infinitely small, being generated by incessant forces; or, finally, when some of these velocities are finite, and some of them infinitely small.

574. To apply this principle, let us consider the impact of two unelastic bodies M and M′, which move in the same direction. Let v and v' represent their velocities before impact, and u the common velocity after impact. The velocity lost by M being equal to its original velocity diminished by that which remains after collision, it will be expressed by $v-u$: in like manner, the velocity lost by M′ will be expressed by $v'-u$. The quantities of motion due to these velocities being such, by the principle of D'Alembert, as to produce an equilibrium, we shall have

$$M(v-u)+M'(v'-u)=0;$$

whence we deduce for the velocity after impact,

$$u=\frac{Mv+M'v'}{M+M'}.$$

When the bodies move in opposite directions, v' will become negative.

575. As a second example, let it be required to determine the circumstances of motion of two bodies M and M′, which

rest on two inclined planes AB and AC (*Fig.* 201) having a common altitude, and are connected by a thread MEM′, passing over a fixed pulley.

If the vertical line Mg, drawn through the centre of gravity of the body M, be supposed to represent the intensity of the force of gravity; the component of the force in the direction of the plane will be represented by MR; this component will alone tend to urge the body down the plane: its value will be expressed by

$$g \times \cos \mathrm{RM}g = g \, . \cos \mathrm{BAD} = g \frac{\mathrm{AD}}{\mathrm{AB}}.$$

In like manner, the component of gravity, which tends to cause the descent of the body M′ on the plane AC, will be expressed by $g\frac{\mathrm{AD}}{\mathrm{AC}}$.

Let the lines AD, AB, and AC be denoted by h, l, and l' respectively; the incessant forces exerted upon the bodies will then be

$$\frac{gh}{l}, \text{ and } \frac{gh}{l'}.$$

But if we suppose the motion to take place in the direction M′EM, and the velocities to be reckoned as positive in this direction, the force $\frac{gh}{l'}$, which is opposed to the motion, must be regarded as negative; and the incessant forces will therefore be expressed by

$$\frac{gh}{l}, \text{ and } -\frac{gh}{l'}.$$

The general expression for the value of an incessant force being

$$\varphi = \frac{dv}{dt},$$

we have

$$dv = \varphi dt:$$

hence, the velocities imparted to the bodies in the time dt, when they are unconnected, will be expressed by

$$\frac{gh}{l}dt, \quad -\frac{gh}{l'}dt;$$

and the quantities of motion due to these velocities will be

$$Mg\frac{h}{l}dt,\quad -M'g\frac{h}{l'}dt.$$

But the bodies being supposed connected by a thread of invariable length, if M should descend through any distance on the plane AB, M′ will necessarily ascend through an equal distance on the plane AC; or, in other words, the velocities of the bodies at any instant will be equal to each other. Denoting by v their common velocity at the end of the time t, the effective velocities communicated to them in the succeeding instant dt, will be expressed by dv, and the effective quantity of motion imparted in the same time, will therefore be

$$(M+M')dv.$$

By the principle of D'Alembert, this quantity of motion when applied in a contrary direction, will produce an equilibrium with the quantities of motion impressed on the bodies: hence, the sum of these quantities of motion will be equal to zero, or

$$-(M+M')dv+Mg\frac{h}{l}dt-\frac{M'gh}{l'}dt=0 \ldots\ldots (325):$$

from which we deduce

$$dv=\frac{M\frac{h}{l}-M'\frac{h}{l'}}{M+M'}.gdt;$$

and by integration,

$$v=\frac{M\frac{h}{l}-M'\frac{h}{l'}}{M+M'}.gt+C \ldots\ldots (325\ a):$$

or, if we denote by G the coefficient of t, we shall have

$$v=Gt+C \ldots\ldots (326).$$

Let x represent the distance OK of the body M from the point O, the origin of the spaces, at the end of the time t; the general expression for the velocity gives

$$v=\frac{dx}{dt};$$

and therefore,

$$\frac{dx}{dt}=Gt+C;$$

from which, by integration, we obtain

$$x=\tfrac{1}{2}Gt^2+Ct+C' \ldots\ldots (327).$$

The formulas (326) and (327) indicate that the circumstances of motion in this system are precisely similar to those which attend the fall of heavy bodies; the only difference consisting in the value of the incessant force, which in the latter case is denoted by g, and in the former by G.

576. If the planes AB and AC be supposed to become vertical, the case will be reduced to that of two weights connected by a cord which passes over a fixed pulley: the quantities h, l, and l' are then equal, and the equations (325 a) and (327) may then be reduced to

$$v=\frac{M-M'}{M+M'}\times gt+C,\quad x=\frac{M-M'}{M+M'}\times\tfrac{1}{2}gt^2+Ct+C' \ldots (327\ a).$$

577. These formulas will serve to explain the principle of Atwood's machine, which is employed for the verification of the laws of constant forces.

This machine consists essentially of, 1°. A fixed pulley, over which passes a very fine flexible thread, having its extremities attached to two equal brass basins; 2°. A vertical graduated scale with a moveable stage to mark the space passed over by the descending basin; and, 3°. A seconds pendulum, by means of which the time of descent may be accurately observed.

When the two basins are loaded with equal weights, they will sustain each other in equilibrio; but if an addition be made to either, it will immediately preponderate, and will produce a motion uniformly varied. Moreover, by rendering the difference M$-$M$'$ of the weights M and M$'$ attached to the extremities of the thread, very small in comparison with their sum M$+$M$'$, the space described and the velocity acquired in a given time which result from equations (327 a) may likewise be rendered small, and the observations will thus become susceptible of great accuracy.

For the purpose of observing the velocity acquired at the end of any time, we give to the additional weight placed in the descending basin the form of a flat bar, and the basin being allowed to pass through a sliding ring attached to the vertical scale, the bar may be removed at any instant during the descent. The equality of the weights in the two basins being restored by the removal of the bar, the motion becomes uniform with the velocity acquired at the instant when the bar was removed.

By comparing the spaces described, the velocities acquired, and the times elapsed, we find that when the basins move from rest under the influence of a constant force, *the velocities are constantly proportional to the times, and that the spaces are proportional to the squares of the times.*

578. For a third example, let it be required to investigate the circumstances of motion of two weights M and M′, which are attached to cords passing around the respective circumferences of a wheel and of its axle.

If we suppose the body M to prevail, and reckon the velocities positive in the direction of its motion, the force of gravity will impress upon the bodies M and M′, in the instant dt, which succeeds the time t, the velocities gdt and $-gdt$; and the quantities of motion impressed will therefore be

$$Mgdt, \text{ and } -M'gdt.$$

But if v and v' represent the velocities of M and M′ at the expiration of the time t, the effective velocities communicated in the succeeding instant dt will be expressed by dv and dv'. Thus, denoting by R and r the radii of the wheel and axle, we shall have

Masses.	Impressed velocities.	Effective velocities.	Distances from the axis.
M	gdt	dv	R,
M′	$-gdt$	dv'	r.

The effective quantities of motion, being applied in directions contrary to those of the motions assumed, will sustain in equilibrio the quantities of motion impressed; and since the equilibrium is maintained through the intervention of the wheel and axle, it is necessary that the sum of the moments with reference to the axis should be equal to zero: hence, we obtain

$$MRgdt-M'rgdt-MRdv-M'rdv'=0 \quad \ldots\ldots \quad (328).$$

This equation containing the two unknown quantities v and v', it will be necessary to discover a second relation between them. For this purpose, we remark that the velocities v and v' bear to each other the constant ratio of $R : r$; thus, we have

$$v : v' :: R : r;$$

or,

$$v' = v\frac{r}{R}:$$

and by differentiating,

$$dv' = \frac{r}{R}dv:$$

substituting this value in equation (328), we find

$$MRgdt - M'rgdt - MRdv - M'\frac{r^2}{R}dv = 0;$$

or, by reduction and transposition,

$$MR^2dv + M'r^2dv = MR^2gdt - M'Rrgdt;$$

whence,

$$dv = \frac{MR^2 - M'Rr}{MR^2 + M'r^2}gdt.$$

Denoting by K the constant coefficient of dt, this equation becomes

$$dv = Kdt;$$

and by integration,

$$v = Kt + C.$$

Replacing v by its value $\frac{dx}{dt}$, and performing a second integration, we find

$$x = \tfrac{1}{2}Kt^2 + Ct + C'.$$

These results indicate that the motion is uniformly varied, the circumstances of the motion being similar to those of a body falling under the influence of the force of gravity.

Of the Motion of a Body about a Fixed Axis.

579. When an impulse is applied to a system of material points connected together in an invariable manner, and subjected to the condition of turning about a fixed axis, which we will suppose to pass through the point A (*Fig.* 202), perpendicular to the plane of the figure, the several particles m,

m', m'', &c. will describe circles mon, $m'o'n'$, $m''o''n''$, &c., the planes of which will be parallel to each other, and perpendicular to the fixed axis; and the arcs described by the several points in the same time will contain the same number of degrees. These arcs being proportional to their radii, the velocities of the several particles will be in the same proportion; so that if we denote by ω the velocity of the particle e, whose distance eA from the axis of rotation is equal to unity, the velocities of the particles m, m', m'', &c., at the distances r, r', r'', &c. from the fixed axis, will be expressed by $r\omega$, $r'\omega$, $r''\omega$, &c. Thus, the effective quantities of motion of the different particles will be represented by

$$mr\omega,\ m'r'\omega,\ m''r''\omega,\ \&c.$$

Let v, v', v'', &c. be the velocities impressed: the corresponding quantities of motion will be expressed by mv, $m'v'$, $m''v''$, &c. It will therefore be necessary, according to the second enunciation of the principle of D'Alembert, that an equilibrium should subsist between the forces mv, $m'v'$, $m''v''$, &c., and $-mr\omega$, $-m'r'\omega$, $-m''r''\omega$, &c.

To establish the conditions of equilibrium between these forces, we will first consider the force mv, and represent it by mf, a portion of its line of direction: from the point f let the perpendicular fh be demitted upon the plane of the section omn, and denote by φ the angle fmh, formed by fm with this plane; by constructing the rectangle hh', the force mv may be resolved into the two components

$mh' = (mv) . \sin \varphi$, parallel to the fixed axis,

$mh = (mv) . \cos \varphi$, situated in the plane omn.

The first of these components will have no tendency to turn the system about the fixed axis; but the second will produce its entire effect in communicating a motion of rotation.

If we represent in like manner by φ', φ'', &c. the angles formed by the directions of the forces $m'v'$, $m''v''$, &c. with the planes $o'm'n'$, $o''m''n''$, &c., the quantities of motion impressed will become

$$mv \cos \varphi,\quad m'v' \cos \varphi',\quad m''v'' \cos \varphi'',\ \&c.$$

These quantities of motion, as well as the quantities $-mr\omega$,

$-m'r'\omega$, $-m''r''\omega$, &c. are situated in planes perpendicular to the fixed axis.

The conditions of equilibrium between these forces will evidently be the same as those which arise when the forces are situated in the same plane; if, therefore, the forces be regarded as situated in the plane of the figure, the conditions of equilibrium will require that the sum of the moments of the forces which tend to turn the system in one direction about the point A, shall be equal to the sum of the moments of those which tend to produce rotation in a contrary direction; or, that the algebraic sum of the moments shall be equal to zero.

But the quantities of motion $-mr\omega$, $-m'r'\omega$, $-m''r''\omega$, &c. being derived from the common motion of the system, they will tend to turn it in the same direction; and since these motions take place in the circumferences of the circles mno, $m'n'o'$, $m''n''o''$, &c., the radii r, r', r'', &c. will represent the perpendiculars demitted from the point A upon their respective directions; consequently, the sum of the moments of the effective quantities of motion, when applied in opposite directions, will be expressed by

$$-mr^2\omega-m'r'^2\omega-m''r''^2\omega-\&c.=-\omega(mr^2+m'r'^2+m''r''^2+\&c.).$$

Let the quantity within the brackets be denoted by $\Sigma(mr^2)$; the sum of these quantities of motion will then be represented by $-\omega\Sigma(mr^2)$.

To determine the value of the sum of the moments of the impressed forces,

$$mv\,.\cos\varphi,\quad m'v'\,.\cos\varphi',\quad m''v''\,.\cos\varphi'', \&c.,$$

let Az (*Fig.* 203) represent the fixed axis, and ml, $m'l'$, $m''l''$, &c. the forces $mv\,.\cos\varphi$, $m'v'\,.\cos\varphi'$, $m''v''\,.\cos\varphi''$, &c. situated in the planes mno, $m'n'o'$, $m''n''o''$, &c., perpendicular to the fixed axis: from the points A, A′, A″, &c., at which the axis intersects the perpendicular planes, let the perpendiculars $Al=p$, $A'l'=p'$, $A''l''=p''$, &c. be demitted upon the directions of the several forces $mv\cos\varphi$, $m'v'\cos\varphi'$, $m''v''\cos\varphi''$, &c.; the moments of these forces will be expressed by

$$mv\cos\varphi\,.\,p,\quad m'v'\cos\varphi'\,.\,p',\quad m''v''\cos\varphi''\,.\,p''.$$

The algebraic sum of these moments will be expressed by

$\Sigma(mv \cos \varphi \,.\, p)$; and hence, by the conditions of equilibrium before enunciated, we shall have

$$\Sigma(mv \,.\, \cos \varphi \,.\, p) - \omega\Sigma(mr^2) = 0.$$

This equation gives the value of the angular velocity

$$\omega = \frac{\Sigma(mv \,.\, \cos \varphi \,.\, p)}{\Sigma(mr^2)} \;\ldots\ldots\; (329);$$

and the motion of the body about the fixed axis will therefore be uniform.

580. When the forces mv, $m'v'$, $m''v''$, &c. are exerted in planes perpendicular to the axis, the angles φ, φ', φ'', &c. become equal to zero, and we have

$$\begin{array}{ll} \sin \varphi = 0, & \cos \varphi = 1, \\ \sin \varphi' = 0, & \cos \varphi' = 1, \\ \sin \varphi'' = 0, & \cos \varphi'' = 1, \\ \&c. & \&c.; \end{array}$$

consequently, the equation (329) reduces to

$$\omega = \frac{\Sigma(mvp)}{\Sigma(mr^2)}.$$

581. If equal velocities be impressed, in parallel directions, upon the several particles m, m', m'', &c., we shall have

$$v = v' = v'' = \&c.$$

and the moments of the quantities of motion impressed will become

$$mvp + m'vp' + m''vp'' + \&c. = v(mp + m'p' + m''p'' + \&c.):$$

the sum of these moments may be represented by $v\Sigma(mp)$, and the equation (329) will be transformed into

$$\omega = \frac{v\Sigma(mp)}{\Sigma(mr^2)} \;\ldots\ldots\; (330).$$

Let a plane AK be now drawn through the axis Az (*Fig.* 204), parallel to the directions of the several forces mv, $m'v'$, $m''v''$, &c.: the perpendiculars p, p', p'', &c. demitted from the points A, A′, A″, &c. upon the directions of these forces, are evidently equal to the perpendiculars mq, $m'q'$, $m''q''$, &c., let fall from the points m, m', m'', &c. upon the plane AK. Let q, q', q'', &c., represent these perpendiculars, and Q the perpendicular demitted from the centre of gravity of the system, upon the plane AK; then, denoting by M the sum of the particles which compose the system, or the entire mass of the body, we shall have, by the property of the centre of gravity,

$$MQ = mq + m'q' + m''q'' + \&c.;$$

and since

$$p = q, \quad p' = q', \quad p'' = q'', \&c.,$$

the preceding equation may be written

$$MQ = mp + m'p' + m''p'' + \&c. = \Sigma(mp).$$

This value being substituted in equation (330), there results

$$u = \frac{vMQ}{\Sigma(mr^2)} \; \ldots\ldots \; (331).$$

582. It may happen that the velocity v has been impressed upon only a limited number of the particles m, m', m'', &c.: then, M will no longer represent the entire mass of the system, but merely the sum of those particles upon which the velocity has been impressed; and Q will express the perpendicular demitted from the centre of gravity of this part of the system upon the plane AK.

The quantity $\Sigma(mr^2)$ is called *the moment of inertia:* the method of determining its value will be explained in the next section.

583. It is frequently necessary to consider the effects produced upon the fixed axis by the application of an impulsive force to any point of the system. For this purpose, let the axis of rotation Az (*Fig.* 205), be assumed as the axis of z, and resolve the impulsion P, which is supposed to be applied at a point O, into two components P′ and P″, which shall be respectively parallel and perpendicular to the plane of x, y. Let the axis of y be then assumed parallel to the direction of P′, and denote the co-ordinates of the point O by a, b, and c: since the force P may be applied at any point in its line of direction, we can always suppose the point of application O to be contained in the plane of x, z: this supposition gives $b=0$.

Instead of regarding the axis as fixed, let such forces be introduced as may be necessary to retain it. These forces will be equal, and directly opposed to the impulsions experienced by the axis, and may in general be reduced to three forces respectively parallel to the axes of x, y, and z. Let X, Y, and Z represent the impulses communicated to the axis, and call AB$=\alpha$, AC$=\beta$.

The particle m will describe a circle parallel to the plane of x, y, and its velocity in the direction of the tangent ml will be expressed by $r\omega$ (Art. 579): the cosines of the angles formed by this direction with the axes of x and y respectively, will be $\frac{y}{r}$ and $-\frac{x}{r}$; hence, the effective quantity of motion of the particle m will be $mr\omega$, and its components in the direction of the axes of x and y will be $my\omega$ and $-mx\omega$: the same remarks apply to the other particles m', m'', m''', &c.

But, by the principle of D'Alembert, an equilibrium will subsist between the effective forces and the force P, the latter being applied in a contrary direction; thus, we shall have

Forces.	Components parallel to axes of x	y	z	Co-ordinates of points of application parallel to x	y	z
$-P$	0	$P\cos\varphi$	$-P\sin\varphi$	a	0	c,
X	X	0	0	0	0	α,
Y	0	Y	0	0	0	β,
Z	0	0	Z	0	0	0,
$mr\omega$	$my\omega$	$-mx\omega$	0	x	y	z,
$m'r'\omega$	$m'y'\omega$	$-m'x'\omega$	0	x'	y'	z',
&c.		&c.			&c.	

The general equations (66) and (67), which express the conditions of equilibrium of forces lying in different planes, and acting upon various points of a body, may be written under the form

$$\Sigma(X)=0, \quad \Sigma(Xy-Yx)=0,$$
$$\Sigma(Y)=0, \quad \Sigma(Zx-Xz)=0,$$
$$\Sigma(Z)=0, \quad \Sigma(Yz-Zy)=0;$$

and when applied to the system under consideration, will give

$$X+\omega\Sigma(my)=0,$$
$$Y+P\cos\varphi-\omega\Sigma(mx)=0,$$
$$Z-P\sin\varphi=0;$$
$$\omega\Sigma(mr^2)-P\cos\varphi a=0,$$
$$X\alpha+\omega\Sigma(myz)+P\sin\varphi a=0,$$
$$Y\beta+P\cos\varphi c-\omega\Sigma(mxz)=0.$$

Let M represent the mass of the body, $x_{,}$ $y_{,}$ and $z_{,}$ the co-ordinates of its centre of gravity, and Mv the quantity of motion which the force P is capable of communicating: these six equations will be reduced to

$$\left.\begin{array}{l} X=-\omega My_{,} \\ Y=\omega Mx_{,}-Mv\,.\cos\varphi \\ Z=Mv\,.\sin\phi \end{array}\right\}\ldots\ldots(331\;a);$$

$$\left.\begin{array}{l} \omega\Sigma(mr^2)=Mv\,.\cos\phi\,.\,a \\ X\alpha=-\omega\Sigma(myz)-Mv\,.\sin\varphi\,.\,a \\ Y\beta=\omega\Sigma(mxz)-Mv\,.\cos\varphi\,.\,c \end{array}\right\}\ldots\ldots(331\;b).$$

From the fourth equation we deduce the value of ω, which being substituted in the first and second, the values of X and Y become known: the third determines the value of Z, and the fifth and sixth give the co-ordinates α and β of the points B and C, at which the forces X and Y are applied. The solution of the problem is therefore complete.

When we wish to communicate the impulse P in such a manner that the axis shall receive no shock, we make X, Y, and Z equal to zero. This supposition reduces the equations (331 a) and (331 b) to the following forms:

$$\begin{array}{ll} y_{,}=0, & \omega\Sigma(mr^2)=Mva, \\ \omega x_{,}=v, & \Sigma(myz)=0, \\ \sin\varphi=0, & \Sigma(mxz)=Mx_{,}c. \end{array}$$

The third equation indicates that the direction of the impulse must be parallel to the plane of x, y; the first, that the centre of gravity of the body must lie in the plane of x, z, perpendicular to which the impulse is applied; the second determines the angular velocity ω; and the fourth and sixth make known the values of the co-ordinates a and c of the point O. The point O is then called the *centre of percussion*, which may be defined to be *that point in the plane passing through the centre of gravity and the axis of rotation, at which an impulse must be applied in a direction perpendicular to this plane, in order that the axis may receive no shock.*

584. The equation $\Sigma(myz)=0$ expresses a relation which is evidently dependent on the figure of the body and the position of the axis of rotation. This relation will exist only in particular cases, and it therefore follows that a body

retained by a fixed axis will not necessarily have a centre of percussion.

585. The distance of the centre of percussion from the axis of rotation being equal to the absciss AN$=a$, its value will be

$$a=\frac{\omega\Sigma(mr^2)}{Mv}=\frac{\Sigma(mr^2)}{Mx_{,}}.$$

586. Although the axis will receive no impulse at the instant of impact, yet the motion of rotation will immediately give rise to centrifugal forces which will exert a pressure upon the axis.

Of the Moment of Inertia.

587. The moment of inertia being the sum of the products formed by multiplying each material point of a system by the square of its distance from a fixed axis, it has been represented in the preceding section by $\Sigma(mr^2)$. In this expression, we may replace the particle m by dM, the element of the mass; and the moment of inertia will then result from the integration of an expression of the form $\int r^2 dM$.

588. For example, let it be required to determine the moment of inertia of a material right line CB (*Fig.* 206), with reference to an axis AZ perpendicular to the plane CAB.

Let AB$=h$ represent the perpendicular demitted from the point A upon the right line, and BP$=x$ the distance of a point P assumed arbitrarily on this line, from the point B: we shall have

$$PA^2=h^2+x^2.$$

This expression being multiplied by the differential of the mass, the integral of the product will express the moment of inertia. The volume, in the present case, being a right line, the element of the volume will be represented by the infinitely small difference dx between two consecutive abscisses BP$=x$ and BP′$=x+dx$; and the element of the mass dM will therefore be expressed by dx multiplied by the density D, or by Ddx. Thus, by multiplying h^2+x^2 by Ddx, and integrating, we obtain for the expression of the moment of inertia of the right line,

$$\int D(h^2+x^2)dx = D(h^2x+\tfrac{1}{3}x^3)+C.$$

In the present disposition of the figure, the integral should be taken between the limits of the point B, where $x=0$, and the point C, at which $x=a$; the moment of inertia thus becomes

$$\left(h^2a+\frac{a^3}{3}\right)D.$$

In effecting this integration, we have regarded the line as homogeneous, or the density D as constant: but if the different parts of the line be supposed unequally dense, the quantity D will be variable, and may in general be regarded as a function of x. The form of this function will depend on the law according to which the density is supposed to vary.

589. When the body is homogeneous, it is frequently convenient to regard the density as equal to unity; and the factor D is then replaced by 1, in the general expression for the moment of inertia. Having determined the moment of inertia of a body whose density is equal to unity, we can determine that of a similar body whose density is equal to D, by simply multiplying the former moment by the density D. In the succeeding examples, we shall regard the density as equal to unity.

590. As a second example, we will determine *the moment of inertia of the area of a circle* CBD (*Fig.* 207), *with reference to the axis* AZ *passing through its centre, and perpendicular to its plane.*

Let m represent a point in the plane of the circle, at a distance $mA=x$ from the fixed axis: the areas of the circles described with the radii x and $x+dx$ will be expressed respectively by

$$\pi x^2, \text{ and } \pi(x+dx)^2;$$

and the difference between these areas, by neglecting the infinitely small quantities of the second order, will be $2\pi x\,.\,dx$. This expression will represent an elementary ring, every point of which will be at the distance x from the axis: hence, by multiplying this element by x^2, we shall obtain $2\pi x^3dx$ for the differential of the moment of inertia. Taking the integral from $x=0$ to $x=r$ we shall find $\frac{1}{2}\pi r^4$ as the moment of inertia of the area of a circle whose radius is denoted by r.

591. *Let it be required to determine the moment of inertia of a sphere with reference to an axis passing through its centre.* If the sphere be cut by a plane EE′ perpendicular to the fixed axis AB (*Fig.* 208), the section will be a circle whose centre will be found at the point D. Denote by x the absciss AD of this section, and by y the ordinate DE, or the radius of the section. The moment of inertia of the area of this circle taken with reference to the axis AB, will be expressed (Art. 590) by

$$\tfrac{1}{2}\pi y^4\,;$$

and if this expression be multiplied by $dx=\text{DD}'$, the product,

$$\tfrac{1}{2}\pi y^4 dx,$$

will express the moment of inertia of the elementary volume EE′F′F bounded by parallel planes drawn through the consecutive points D and D′. The integral of this expression, being taken between the limits $x=0$ and $x=\text{AB}=2r$, will give the moment of inertia of the entire sphere.

But by the property of the circle, we have

$$y^2=2rx-x^2\,;$$

and therefore,

$$\begin{aligned}\int \tfrac{1}{2}\pi y^4 dx &= \tfrac{1}{2}\pi\int(2rx-x^2)^2 dx\\ &= \pi\int(2r^2x^2-2rx^3+\tfrac{1}{2}x^4)dx\,;\end{aligned}$$

or,

$$\int \tfrac{1}{2}\pi y^4 dx=\pi x^3(\tfrac{2}{3}r^2-\tfrac{1}{2}rx+\tfrac{1}{10}x^2)+\text{C}.$$

The constant C will be equal to zero, since the moment is zero when $x=0$: and by making $x=2r$, we obtain for the moment of the whole sphere,

$$\tfrac{8}{15}\pi r^5.$$

These examples are sufficient to explain the manner in which the determination of the moment of inertia is reduced to a simple problem of the integral calculus.

592. When the moment of inertia of any body with reference to an axis passing through its centre of gravity has been determined, its moment with respect to a parallel axis is readily found.

For let GF and CK (*Fig.* 209) represent two parallel axes, the first of which passes through G, the centre of gravity of a body: let the origin be assumed at the point G, the line GF being the axis of z. Through a point m, assumed arbitrarily within the limits of the body, let the plane mKF be drawn, parallel to the plane of x, y; this plane will cut the axes GF and CK at two points F and K, and the distances of the point m from these axes will be represented respectively, by the right lines mK and mF, which we shall denote by r and r'. From the point m let the perpendicular mE be demitted upon the plane of x, y; the triangles ECG, mKF will be equal in all respects, and the sides of the former may therefore be substituted for those of the latter. Denote by

α and β, the co-ordinates GD and DC of the point C,
x and y, the co-ordinates GP and PE of the point E,
a, the distance between the axes:

we shall have

$$GC^2 = GD^2 + DC^2, \quad GE^2 = GP^2 + PE^2,$$

or,

$$a^2 = \alpha^2 + \beta^2, \quad r'^2 = x^2 + y^2 \; \ldots\ldots \; (332).$$

Again, the right line CE passing through points whose co-ordinates are x and y, α and β; the value of $CE = r$ will result from the equation

$$r^2 = (x-\alpha)^2 + (y-\beta)^2,$$

or, by developing the terms of the second member,

$$r^2 = x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2;$$

and reducing by means of equations (332), we obtain

$$r^2 = r'^2 - 2\alpha x - 2\beta y + a^2;$$

multiplying by dM and integrating, we have

$$\int r^2 dM = \int r'^2 dM - 2\alpha \int x dM - 2\beta \int y dM + a^2 \int dM \; \ldots\ldots \; (333).$$

The expressions $\int x dM$ and $\int y dM$ which enter into this equation, are equal to zero; for, let x and y represent the co-ordinates of the element dM of the mass M; the moments of this element with reference to the planes of x, z and y, z will be $y dM$ and $x dM$: hence, the co-ordinates $x_{,}$ and $y_{,}$ of the centre of gravity of the mass M will be determined by the equations

$$Mx_{,} = \int x dM, \quad My_{,} = \int y dM.$$

But in the present instance, the centre of gravity is situated in the axis of z; and the co-ordinates $x_{,}$ and $y_{,}$ are therefore equal to zero: hence,

$$\int x dM=0, \quad \int y dM=0.$$

Reducing equation (333) by means of these values and substituting M for its equal $\int dM$, we shall obtain

$$\int r^2 dM=\int r'^2 dM+Ma^2 \ldots\ldots (334).$$

The expression $\int r'^2 dM$ being the moment of inertia with reference to the axis passing through the centre of gravity, we conclude that when the value of this moment has been found, that of the moment of inertia $\int r^2 dM$, taken with reference to a parallel axis, may be immediately determined, by adding to the former the product of the mass of the body by the square of the distance between the two axes.

The equation (334) may be written under the form

$$\int r^2 dM=M\left(\frac{\int r'^2 dM}{M}+a^2\right),$$

and this expression may be simplified by putting $\frac{\int r'^2 dM}{M}=k^2$.

Adopting this notation, the moment of inertia taken with reference to any axis will be expressed by the formula

$$\int r^2 dM=M(k^2+a^2).$$

Of the Motion of a Body about a Fixed Axis when acted upon by Incessant Forces.

593. Let us now suppose that the several material points of a system which is retained by a fixed axis Az (*Fig.* 210), are acted upon by incessant forces: each particle m will describe about the fixed axis, the arc of a circle mno, the plane of which will be perpendicular to this axis, and will intersect it at a point C. Let ϕ denote the incessant force acting upon the particle m, and δ the angle TmP formed by its direction with the tangent to the circle mno at the point m. The force ϕ may be resolved into three components; one parallel to the fixed axis, which will have no tendency to turn the body about this axis; a second directed along the radius

mC, which will be destroyed by the resistance of the axis; and a third coinciding in direction with the element of the curve described by the particle m: this last component will be expressed by $\phi \cos \delta$, and will be the only portion of the force ϕ which tends to turn the system about the axis Az.

Let ω represent the angular velocity of the system at the expiration of the time t, and r the distance Cm of the particle m from the axis of rotation: the absolute velocity of m, at the end of the time t, will be expressed by $r\omega$ (Art. 579), and in the succeeding instant dt, this velocity will be increased or diminished by the action of the incessant force.

If the particle m were unconnected with the other particles, the force $\phi \cos \delta$ would communicate to it in the instant dt, the velocity represented by $\phi \cos \delta . dt$; consequently, the velocity of the particle m, at the expiration of the time $t+dt$, would be expressed by

$$r\omega + \phi \cos \delta . dt;$$

but this particle being connected with the other parts of the system, its effective velocity at the end of the time $t+dt$ will actually be represented by

$$r\omega + rd\omega;$$

and the effective quantity of motion of the particle m will be $(r\omega + rd\omega)m$.

The same remarks being applicable to the other particles which compose the system, it is necessary that the quantities of motion impressed, or

$$\Sigma[(r\omega + \phi \cos \delta . dt)m]$$

should, by the principle of D'Alembert, sustain in equilibrio the effective quantities of motion

$$\Sigma[(r\omega + rd\omega)m],$$

the latter being applied in directions contrary to those of the motions assumed.

But, in order that an equilibrium may subsist between these two sets of forces, it is necessary that the sum of the moments of the several forces taken with reference to the fixed axis, shall be equal to zero: and since these forces are exerted in the directions of the elements of the circles described by the material points, the radii of these circles will represent the perpendiculars demitted upon the directions of the several forces. The equation of the moments will thus become

$$\Sigma[(r^2\omega+r\varphi\cos\delta\,.\,dt)m]-\Sigma[(r^2\omega+r^2d\omega)m]=0;$$

or, by reduction,

$$\Sigma(r^2d\omega\,.\,m)=\Sigma(r\varphi\cos\delta\,.\,dt\,.\,m)\;\ldots\ldots\;(335).$$

The quantities dt and $d\omega$ being the same in all the terms of this equation, they may be placed without the sign Σ; and when the number of terms is regarded as infinite, the value of each being infinitely small, the character Σ may be replaced by the integral sign $\int$, and the particle m by dM, the differential of the entire mass: thus we shall have

$$dt\int r\,.\,\varphi\cos\delta\,.\,d\text{M}=d\omega\int r^2d\text{M};$$

from this equation, we deduce

$$\frac{d\omega}{dt}=\frac{\int r\,.\,\varphi\cos\delta\,.\,d\text{M}}{\int r^2d\text{M}}\;\ldots\ldots\;(336).$$

To complete the integrations here indicated, it is necessary to know the positions of the elements which compose the body, and the directions and intensities of the incessant forces exerted upon each particle. These particulars will be examined in the following section.

Of the Compound Pendulum.

594. The compound pendulum, represented in *Fig.* 211, is composed of a body, or a system of material points, connected together in an invariable manner, and supported by a horizontal axis KL. When the body is turned around this axis, the points m, m', m'', &c. describe arcs of circles mn, $m'n'$, $m''n''$, &c.; the centres of these circles are situated in the axis KL, and their planes are perpendicular to it.

595. The motion of the pendulum being referred to three rectangular axes, let the axis of z be supposed to coincide with the horizontal line Cz (*Fig.* 212), about which the body turns, and the axis of x to be vertical; the plane of z, y will then be horizontal. If we suppose the incessant force exerted upon each particle to be that of gravity, we shall have

$$\varphi=\varphi'=\varphi''=\&\text{c.}=g.$$

The direction of the force which solicits a particle m, being parallel to the axis of x, the intensity of this force may be represented by a portion mg of a vertical line; the angle δ will be equal to Tmg; and if the perpendicular mD be demitted upon the axis of x, the angles CmD and Tmg will be equal to each other, being each the complement of the angle TmD: hence, CmD$=\delta$; and consequently, the equation

$$mD=Cm\times\cos CmD$$

will become

$$mD=Cm\,.\cos\delta;$$

or,

$$y=r\,.\cos\delta.$$

The values of cos δ and φ being substituted in equation (336), we obtain

$$\frac{d\omega}{dt}=\frac{\int gydM}{\int r^2dM};$$

or, since g is constant,

$$\frac{d\omega}{dt}=\frac{g\int ydM}{\int r^2dM}.$$

The expression ydM represents the moment of the elementary mass dM taken with reference to the plane of x, z; if, therefore, we denote by $y_{,}$ the distance of the centre of gravity of the entire mass M from the same plane, we may replace $\int ydM$ by $My_{,}$, and the preceding equation will then become

$$\frac{d\omega}{dt}=\frac{gMy_{,}}{\int r^2dM}\ \ldots\ldots\ (337):$$

and since $\int r^2dM$ expresses the moment of inertia with reference to the axis Cz, this moment may be represented (Art. 592) by $M(k^2+a^2)$. Substituting this value in equation (337), we find

$$\frac{d\omega}{dt}=\frac{gy_{,}}{k^2+a^2}\ \ldots\ldots\ (338).$$

596. It has been shown (Art. 592), that the quantity a in the expression $M(a^2+k^2)$ represents the distance CG (*Fig.* 209) between the axis CK and the parallel axis GF passing through the centre of gravity. But, by the motion of the system, the centre of gravity describes a circle having its radius CG$=a$ (*Fig.* 213), and its plane xCL perpendicular to

the axis CK; hence, the ordinate DG will represent the quantity $y_{,}$ and we shall have from the property of the circle,

$$y_{,}=\surd(2ax_{,}-x_{,}^2).$$

Again, if s denote the arc described by the point G, the velocity of this point will be expressed by $\frac{ds}{dt}$: but this velocity will also be expressed by $a\omega$ (Art. 579). Hence, we shall have

$$a\omega=\frac{ds}{dt};$$

and consequently,

$$\omega=\frac{ds}{adt}.$$

The values of ω and $y_{,}$ being substituted in equation (338), convert it into

$$\frac{d^2s}{adt^2}=\frac{g\surd(2ax_{,}-x_{,}^2)}{k^2+a^2}.$$

597. If we multiply each member of this equation by $2ads$, the first member will become an exact differential, and we shall obtain by integration,

$$\frac{ds^2}{dt^2},\quad \text{or } v^2=\int\frac{2ag}{k^2+a^2}ds\surd(2ax_{,}-x_{,}^2)\ldots\ldots(339).$$

The integral of the second member can only be obtained after eliminating one of the two variables which it contains: this may be effected by means of the equations

$$ds=\surd(dx_{,}^2+dy_{,}^2),\quad y_{,}=\surd(2ax_{,}-x_{,}^2);$$

and by proceeding as in Art. 465, we find

$$ds=\frac{-adx_{,}}{\surd(2ax_{,}-x_{,}^2)};$$

substituting this value in equation (339), we have

$$v^2=-\int\frac{2a^2g}{k^2+a^2}dx_{,};$$

whence, by integration,

$$v^2=-\frac{2a^2gx_{,}}{k^2+a^2}+\text{C}\ldots\ldots(340).$$

To determine the value of the constant C, let EB$=b$ represent the value of $x_,$ at the instant when $v=0$; the supposition of $v=0$ and $x=b$ gives

$$C=\frac{2a^2gb}{k^2+a^2};$$

and the equation (340) will therefore become

$$v^2,\ \text{ or } \frac{ds^2}{dt^2}=\frac{2a^2g}{k^2+a^2}(b-x_,);$$

whence,

$$dt=\frac{ds}{\sqrt{\left(\frac{2a^2g}{k^2+a^2}\cdot(b-x_,)\right)}} \quad \ldots\ldots (341).$$

This equation can be readily integrated when the oscillations are performed through very small arcs, as usually happens; for, by replacing ds by its value $-\frac{adx_,}{\sqrt{(2ax_,)}}$ obtained on the supposition that $x_,$ may be neglected as exceedingly small in comparison with $2a$, in the expression

$$ds=\frac{-adx_,}{\sqrt{(2ax_,-x_,^2)}};$$

the equation (341) becomes

$$dt=-\frac{\frac{1}{2}dx_,}{\sqrt{\left(\frac{ag}{k^2+a^2}(b-x_,)x_,\right)}},$$

which may be written under the form

$$dt=-\tfrac{1}{2}\sqrt{\left(\frac{k^2+a^2}{ag}\right)}\times\frac{dx_,}{\sqrt{[(b-x_,)x_,]}} \quad \ldots\ldots (342).$$

598. By comparing this equation with the equation (228), it will appear that they differ only by the constant factor, which in the former is $\frac{1}{2}\sqrt{\left(\frac{k^2+a^2}{ag}\right)}$, and in the latter $\frac{1}{2}\sqrt{\frac{a}{g}}$. Hence, the integral of (342) may be immediately obtained from that of (228), the constants being determined by the same condition, that when $t=0$, $x_,=b$. Consequently, if we denote by l the length of a simple pendulum, or if we replace $\frac{a}{g}$ in equation (228) by $\frac{l}{g}$, and determine l by the condition

$$\frac{l}{g}=\frac{k^2+a^2}{ag},$$

the simple pendulum and the compound pendulum will perform their oscillations in the same time. The preceding equation gives

$$l=\frac{k^2+a^2}{a}.$$

Thus, by means of this formula we can always find the length of the simple pendulum which will perform its oscillations in the same time as a given compound pendulum.

599. If, at the distance l from the axis of suspension AB, a line EF (*Fig.* 214) be drawn parallel to the axis AB, this parallel will enjoy the property, that all points contained in it will perform their oscillations in the same time as though they were unconnected with the other points of the body. When the line EF is contained in the plane passing through the axis of suspension AB and the centre of gravity of the body, this line is called the *axis of oscillation*, and its several points are called *centres of oscillation*.

600. *The axes of suspension and oscillation are reciprocal;* that is to say, if we take the axis of oscillation EF (*Fig.* 214) as a new axis of suspension, the corresponding axis of oscillation will coincide with the original axis of suspension.

To demonstrate this property, we resume the expression for CD, the distance between the axes of suspension and oscillation given in Art. 598,

$$l=\frac{a^2+k^2}{a}\ldots\ldots(343).$$

If we then assume the line EF as an axis of suspension, and represent by l' and a' the corresponding distances of the centres of oscillation and gravity from this axis, we shall have by the nature of the centre of oscillation,

$$l'=\frac{a'^2+k^2}{a'}\ldots\ldots(344).$$

And since the equation (343) indicates that the distance l exceeds a, it follows that the centre of gravity will be situated between the axes of suspension and oscillation. We shall therefore have the following relation,

$$a+a'=l,$$

or,

$$a'=l-a.$$

By means of this value, the equation (344) becomes

$$l'=\frac{(l-a)^2+k^2}{l-a} \ldots\ldots (345).$$

Again, from equation (343) we have

$$l-a=\frac{k^2}{a};$$

and the value of l' may therefore be changed into

$$l'=\frac{\frac{k^4}{a^2}+k^2}{\frac{k^2}{a}};$$

or, by reduction,

$$l'=\frac{k^2}{a}+a=l:$$

consequently, when the line EF is taken as the axis of suspension, the axis of oscillation KH is situated at a distance MX from the line EF, precisely equal to that which separates the axes AB and EF.

601. The equation (343) gives

$$a(l-a)=k^2;$$

and by replacing $l-a$ by its value a', we have

$$aa'=k^2:$$

but the value of k^2, which is dependent on the moment of inertia taken with reference to an axis passing through the centre of gravity, and parallel to the axis AB, will remain constant so long as the direction of the axis remains unchanged: hence it appears that if the body be caused to oscillate about any axis parallel to AB, and at a distance from the centre of gravity represented by a, the corresponding axis of oscillation will be found at a distance a' from the centre of gravity; thus the value of $a+a'$, or the length of the equivalent simple pendulum, will be the same as when the oscillations were performed about the axis AB. A similar remark is applicable to all those axes parallel to AB which are situated at a distance a' from the centre of gravity. If, therefore, the body be suspended successively from any number of axes parallel to AB, and at a distance from the centre of gravity equal to a or a', the times of oscillation about such axes will be equal to each other.

These parallel axes of suspension about which the oscillations are performed in equal times, will evidently be found in the surfaces of two cylinders having a common axis passing through the centre of gravity.

602. The expression for the distance l between the axes of suspension and oscillation may be put under the form

$$l=\frac{M(a^2+k^2)}{Ma}=\frac{\Sigma(mr^2)}{Ma};$$

and since this value is precisely equal to that which was obtained for the distance of the centre of percussion from the axis of rotation (Art. 585), it appears that the centre of percussion, when it exists, will be found upon the axis of oscillation.

Of the Motions of a Body in Space when acted upon by Impulsive Forces.

603. In the preceding sections, the circumstances of motion of a body retained by a fixed axis have been alone discussed; it now becomes necessary to consider the motions of a body in space when unconnected with fixed objects.

Let m, m', m'', &c. represent material points composing a system whose several particles are unconnected, and let v, v', v'', &c. represent the velocities respectively impressed upon these particles in directions parallel to each other: it is required to determine the motion of the common centre of gravity of the system.

If a plane be passed through the primitive position of the centre of gravity parallel to the common direction in which the impulses are applied, the sum of the moments of the particles m, m', m'', &c., taken with reference to this plane, will be equal to zero at the commencement of the motion;

and it is likewise evident that this sum will remain equal to zero during the motion, since the distances of the bodies from the assumed plane remain invariable. Hence, the motion of the centre of gravity will be confined to this plane; and since the same may be said of any other plane drawn through the primitive position of the centre of gravity and parallel to the direction of the motions, it follows that the centre of gravity will continue in each of these planes, or in their line of intersection; and we therefore conclude that *the motion of the centre of gravity of such a system is rectilinear, and parallel to the direction of the motions of its several parts.*

Let a plane be drawn perpendicular to the direction in which the bodies move, and represent the distances of the several bodies from this plane at the commencement of the motion, by S, S′, S″, &c.: their distances, at the expiration of the time t, will be expressed by

$$S+vt, \quad S'+v't, \quad S''+v''t, \text{ \&c.}$$

If a and $x_{\prime}$ represent the distances of the centre of gravity of the system from the perpendicular plane, at the commencement of the motion, and at the end of the time t, we shall have, by the property of the centre of gravity,

$$mS+m'S'+m''S''+\text{\&c.}=(m+m'+m''+\text{\&c.})a,$$

$$m(S+vt)+m'(S'+v't)+m''(S''+v''t)+\text{\&c.}=$$
$$(m+m'+m''+\text{\&c.})x_{\prime};$$

and by subtraction, we obtain

$$(m+m'+m''+\text{\&c.})(x_{\prime}-a)=(mv+m'v'+m''v''+\text{\&c.})t:$$

hence, it appears that the space passed over by the centre of gravity is proportional to the time, or *the motion of the centre of gravity is uniform.*

It is to be understood that those velocities are regarded as negative, whose directions are opposite to such as we consider positive.

604. The preceding equation may be written under the form

$$(m+m'+m''+\text{\&c.})\frac{x_{\prime}-a}{t}=mv+m'v'+m''v''+\text{\&c.};$$

the expression $\frac{x_{\prime}-a}{t}$ represents the velocity of the centre of gravity, and is independent of the positions of the particles m, m', m'', &c., to which the quantities of motion mv, $m'v'$, $m''v''$, &c. are respectively applied: it follows, therefore, that if we suppose a mass M equal to the sum of the masses m, m', m'', &c. to be concentrated at the centre of gravity, the quantity of motion of this mass will be equal to the sum of the quantities of motion in the entire system.

We also conclude, that *the centre of gravity will have the same motion as though the several masses m, m', m'', &c. were concentrated in this point, and the several forces applied immediately to it in directions parallel to those along which they were originally applied.*

605. When the forces applied to the different particles are not parallel, they may be resolved into components parallel to three rectangular axes, and since the effects produced by each system of parallel components will be independent of the other two systems, it may in like manner be shown that the motion of the centre of gravity parallel to each of the axes will be uniform, and equal to that which would be produced by concentrating the masses at the centre of gravity, and applying the several forces directly to that point.

606. Let the several masses be now supposed connected in an invariable manner, the same property will be equally true. For, let mv, $m'v'$, $m''v''$, &c. represent the quantities of motion impressed upon the particles m, m', m'', &c., and let each of these quantities of motion be resolved into components mu and mU, &c., the first of which shall be the effective quantity of motion retained by the particle, the second being destroyed by the mutual connexion of the parts of the system: then, since the quantities of motion mu, $m'u'$, $m''u''$, &c., communicated to the masses m, m', m'', &c., produce their full effects, these masses will move under their influence, in the same manner, whether we regard them as free or connected.

Hence, it appears that the centre of gravity of the system will move in the same manner as though the quantities of motion mu, $m'u'$, $m''u''$, &c, were applied directly to it, The quantities of motion mU, m'U', m''U'', &c. being such as to destroy each other when applied to the different points m, m',

m''', &c., they must (Arts. 54 and 130) destroy each other when applied to the centre of gravity.

But the two systems mu, $m'u'$, $m''u''$, &c., mU, $m'U'$, $m''U''$, &c., may be replaced by the original system mv, $m'v'$, $m''v''$, &c., and we therefore conclude that *the centre of gravity will have the same motion as though the several masses had been concentrated at that point, and the original quantities of motion mv, $m'v'$, $m''v''$, &c. impressed immediately upon it.*

607. If an impulse P be communicated to any point of a body in a direction not passing through the centre of gravity, this centre will assume a motion precisely equal to that which would have been produced by the direct application of the force to it. But a motion of rotation will also be communicated to the body; for, if an equal force Q (*Fig.* 215) be applied to the centre of gravity in a parallel and opposite direction, the joint action of the two forces P and Q will maintain the centre of gravity at rest. From the centre of gravity G demit the perpendicular GA upon the direction of the force P, and lay off on the opposite side of the point G a distance GB=AG. Let the force Q be then resolved into two components, each equal to $\frac{1}{2}$Q or $\frac{1}{2}$P, applied at the points A and B. The forces P and $\frac{1}{2}$Q applied at the point A, and acting in contrary directions, will have a resultant equal to $\frac{1}{2}$P: thus the body will be acted on by two forces each equal to $\frac{1}{2}$P, acting at the distance AG=BG from the centre of gravity, and tending to turn the body about that point. And since the point G may be regarded as fixed, the two forces will have the same effect to turn the body about that point as the single force P acting at A. The effect of the force Q will be simply to destroy the motion of translation, without affecting the motion of rotation.

Hence we conclude, that *when a body receives an impulse in a direction which does not pass through the centre of gravity, that centre will assume a motion of translation as though the impulse were applied immediately to it; and the body will likewise have a motion of rotation about the centre of gravity, as though that point were immoveable.*

608. The circumstances of motion of a body which is divided symmetrically by a plane passing through the direc-

tion of the impulse can now be readily determined. For, the motion of translation of the centre of gravity will be similar to that of a material point to which an impulse is applied; and the motion of rotation being precisely the same as that which would take place if a fixed axis passed through the centre of gravity, perpendicular to the dividing plane, it will merely be necessary to apply the results obtained in Arts. 581 and 582.

Let Mv represent the quantity of motion impressed upon a body whose mass is represented by M (*Fig.* 216), and p the perpendicular distance from the centre of gravity G to the line of direction of the impulse. The centre of gravity will assume a uniform motion with the velocity v, in a direction parallel to that of the impulsive force. The angular velocity will result immediately from equation (331), and will be expressed by

$$\omega=\frac{Mvp}{Mk^2}=\frac{vp}{k^2}.$$

609. The absolute velocity of each point of the body will be compounded of the two velocities of translation and rotation. Thus, the point O, for example, to which the force is applied, has two velocities; a velocity of translation Oi equal to that of the centre of gravity, and a velocity of rotation ih about that point; so that if we assume any point on the line OGC, at a distance a from the centre of gravity, its velocity will be expressed by $v\pm a\omega$: the superior sign applies to those points which are situated upon the same side of the centre of gravity as the point O; and the inferior sign to points situated on the opposite side.

610. If we consider the motion of the point O for an exceedingly short interval of time, the path Oih described by this point, whilst the centre of gravity describes the line GG′, may be regarded as a right line: thus, the line OGC will assume the position hG′C, the point C remaining at rest during this interval. This point is called *the centre of spontaneous rotation*: its position may be determined by the condition that its velocity of rotation shall be equal to that of translation: indeed, whilst the point C would be carried forward over the line CC′ by the motion of translation, it would

be moved backward through the same distance by the motion of rotation: this condition will give the absolute velocity of the point C

$$v-a\omega=0;$$

whence,

$$a=\frac{v}{\omega}=\frac{k^2}{p};$$

and we therefore have

$$\mathrm{OC}=\mathrm{OG}+\mathrm{GC}=p+a=p+\frac{k^2}{p};$$

from which we conclude, that *the centre of spontaneous rotation will coincide with the centre of percussion, if the axis of rotation be supposed to pass through the point O.*

611. When the plane passing through the direction of the impulse and the centre of gravity divides the body into two portions which are not symmetrically situated with respect to this plane, it will usually occur that the axis about which the body revolves will not retain an invariable position. For, the rotatory motion of the body will develop in each particle a centrifugal force, producing a pressure upon the axis; and unless these pressures are such as to destroy each other, the direction of the axis will necessarily be changed.

Of the Motions of a System in Space when acted upon by Incessant Forces.

612. We will next investigate the circumstances of motion in a system whose different particles are acted upon by incessant forces. Let the force acting on a particle m be resolved into three components X, Y, Z, respectively parallel to three rectangular axes; that acting on m' into the three X′, Y′, Z′, &c. Let a, b, and c represent the variable co-ordinates of the centre of gravity referred to the fixed axes, and let three axes be drawn through the centre of gravity, parallel to the fixed axes, and moveable with the system in space. Then, if x, y, z, x', y', z', &c. denote the co-ordinates of the points m, m', m'', &c. referred to the moveable axes; $a+x$, $b+y$, $c+z$, $a+x'$, $b+y'$, $c+z'$, &c. will express the co-ordinates of the same points when referred to the fixed axes.

613. The velocity of the particle m in the direction of the axis of x, at the expiration of the time t, will be expressed by

$$v=\frac{d(a+x)}{dt}=\frac{da+dx}{dt};$$

and in the succeeding instant dt, this velocity would receive the increment $\mathrm{X}dt$, by the action of the incessant force X, if the praticle m were entirely free; but in consequence of the connexion existing between the different parts of the system, the effective velocity communicated to the particle m in the time dt, will be expressed by

$$dv=d\ \frac{da+dx}{dt},$$

and the velocity destroyed in the particle m, by the connexion of the parts of the system, will therefore be

$$\mathrm{X}dt-d\ \frac{da+dx}{dt}.$$

The same remarks being applicable to the velocities parallel to the axes of y and z, we shall have for the quantities of motion destroyed in the particle m, parallel to the three axes,

$$m\left(\mathrm{X}dt-d\ \frac{da+dx}{dt}\right),$$

$$m\left(\mathrm{Y}dt-d\ \frac{db+dy}{dt}\right),$$

$$m\left(\mathrm{Z}dt-d\ \frac{dc+dz}{dt}\right).$$

Similar expressions may in like manner be obtained for the quantities of motion lost by the other particles; and we shall therefore obtain, for the sum of the quantities of motion lost parallel to the axis of x,

$$\Sigma\left[m\left(\mathrm{X}dt-d\ \frac{da+dx}{dt}\right)\right]\ldots\ldots(346);$$

or, by completing the differentiation indicated, regarding dt as constant, we have

$$\Sigma\left[m\left(\mathrm{X}dt-\frac{d^2a+d^2x}{dt}\right)\right].$$

In like manner, the sums of the quantities of motion lost in directions parallel to the axes of y and z, will be expressed by

$$\Sigma\left[m\left(\mathrm{Y}dt-\frac{d^2b+d^2y}{dt}\right)\right]\ldots\ldots(347),$$

$$\Sigma\left[m\left(\mathrm{Z}dt-\frac{d^2c+d^2z}{dt}\right)\right]\ldots\ldots(348)$$

The quantities of motion (346), (347), (348), or the forces capable of producing them, being such as to destroy each other, they must satisfy the general equations of equilibrium (66) and (67), which appertain to a system of forces having various directions and applied to different points of a body.

The equations (66) indicate that the sum of the components parallel to each of the axes will be equal to zero; we shall therefore have for those components parallel to the axis of x

$$\Sigma\left[m\left(\mathrm{X}dt-\frac{d^2a+d^2x}{dt}\right)\right]=0;$$

or, by multiplying by dt, and changing the form of the expression, we have

$$\begin{aligned}0=(m\mathrm{X}+m'\mathrm{X}'+m''\mathrm{X}''+\&c.)dt^2-d^2a(m+m'+m''+\&c.)\\-(md^2x+m'd^2x'+m''d^2x''+\&c.)\ldots\ldots(349).\end{aligned}$$

But, by the nature of the centre of gravity,

$$\left.\begin{aligned}mx+m'x'+m''x''+\&c.=0\\my+m'y'+m''y''+\&c.=0\end{aligned}\right\}\ldots\ldots(349\ a):$$

and by differentiating twice, we find

$$\left.\begin{aligned}md^2x+m'd^2x'+m''d^2x''+\&c.=0\\md^2y+m'd^2y'+m''d^2y''+\&c.=0\end{aligned}\right\}\ldots\ldots(349\ b).$$

The first of these values being substituted in (349), and the mass of the system being denoted by M, there will result

$$\mathrm{M}d^2a=(m\mathrm{X}+m'\mathrm{X}'+m''\mathrm{X}''+\&c.)dt^2,$$

or,

$$\mathrm{M}\frac{d^2a}{dt^2}=\Sigma(m\mathrm{X}):$$

the same being true with respect to the components parallel to the axes of y and z, we shall obtain, for the three first equations expressing the circumstances of motion of the system,

$$\left.\begin{aligned}\mathrm{M}\frac{d^2a}{dt^2}=\Sigma(m\mathrm{X})\\\mathrm{M}\frac{d^2b}{dt^2}=\Sigma(m\mathrm{Y})\\\mathrm{M}\frac{d^2c}{dt^2}=\Sigma(m\mathrm{Z})\end{aligned}\right\}\ldots\ldots(350).$$

These equations serve to determine the motion of the centre of gravity of the mass M; for when integrated, they will express the velocities $\frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt}$ of the centre of gravity, parallel to the three axes.

614. The equations (350) make known a remarkable property of the centre of gravity. For, let the particles m, m', m'', &c. be supposed concentrated at their common centre of gravity, and let the forces mX, mY, mZ, $m'X'$, $m'Y'$, $m'Z'$, &c. be applied directly to that point, parallel to their original directions. These forces may be reduced to three, $MX_{,}$, $MY_{,}$, $MZ_{,}$ the values of which will result from the equations

$$MX_{,}=\Sigma(mX), \quad MY_{,}=\Sigma(mY), \quad MZ_{,}=\Sigma(mZ.)$$

Eliminating the second members of these equations by means of equations (350), we have

$$\frac{d^2a}{dt^2}=X_{,}, \quad \frac{d^2b}{dt^2}=Y_{,}, \quad \frac{d^2c}{dt^2}=Z_{,} \ldots\ldots (351).$$

But when the forces $MX_{,}$, $MY_{,}$, $MZ_{,}$ are applied to the centre of gravity regarded as a material point whose mass is M, the circumstances of its motion are expressed by the equations (180), which are precisely similar to the equations (351); hence, we conclude that *the centre of gravity of the system has the same motion as though the forces were applied directly to that point.*

615. To determine the circumstances of motion of the several particles m, m', m'', &c. with respect to the centre of gravity, we resume the equations (67), which express the conditions that the forces have no tendency to turn the system about either axis: that this may be the case, it is necessary that the sum of the differences of the moments of the components parallel to any two of the axes, as x and y, taken with reference to the corresponding planes of y, z and x, z, should be equal to zero. But if we consider the particle m, the distance of the component X, which acts upon it, from the plane of x, z will be equal to $y+b$, the co-ordinate of the point m, parallel to the axis of y: in like manner, the distance of the force Y from the plane of y, z will be expressed by $x+a$: we shall therefore have, for the difference of the moments,

$$m\left(\mathrm{X}-\frac{d^2a}{dt^2}-\frac{d^2x}{dt^2}\right)(b+y)-m\left(\mathrm{Y}-\frac{d^2b}{dt^2}-\frac{d^2y}{dt^2}\right)(a+x).$$

The same remarks being applicable to the particles m', m'', &c., we shall obtain a similar expression for each. By placing the sum of these expressions equal to zero, as in equation (67), performing the multiplications, and reducing by means of equations (349 *a*) and (349 *b*), we shall obtain

$$b\Sigma(m\mathrm{X})-\mathrm{M}b\frac{d^2a}{dt^2}+\Sigma(my\mathrm{X})-\Sigma\left(my\frac{d^2x}{dt^2}\right)$$
$$-a\Sigma(m\mathrm{Y})+\mathrm{M}a\frac{d^2b}{dt^2}-\Sigma(mx\mathrm{Y})+\Sigma\left(mx\frac{d^2y}{dt^2}\right)=0.$$

This equation admits of simplification; for, if we multiply the first of equations (350) by b, and the second by a, and take their difference, we shall have

$$b\Sigma(m\mathrm{X})-a\Sigma(m\mathrm{Y})-\mathrm{M}b\frac{d^2a}{dt^2}+\mathrm{M}a\frac{d^2b}{dt^2}=0.$$

This relation reduces the previous equation to

$$\Sigma(my\mathrm{X})-\Sigma(mx\mathrm{Y})-\Sigma\left(my\frac{d^2x}{dt^2}\right)+\Sigma\left(mx\frac{d^2y}{dt^2}\right)=0;$$

whence,

$$\Sigma\left(m\,\frac{xd^2y-yd^2x}{dt}\right)=\Sigma[m(\mathrm{Y}x-\mathrm{X}y)dt].$$

The integral of the first member, taken with reference to the time t, is

$$\Sigma\left(m\frac{xdy-ydx}{dt}\right):$$

and by adopting the same process with reference to the other two axes, putting, for brevity,

$$\Sigma[m\textstyle\int(\mathrm{Y}x-\mathrm{X}y)dt]=\mathrm{L},$$
$$\Sigma[m\textstyle\int(\mathrm{Z}x-\mathrm{X}z)dt]=\mathrm{M},$$
$$\Sigma[m\textstyle\int(\mathrm{Z}y-\mathrm{Y}z)dt]=\mathrm{N},$$

we shall obtain the three equations of motion

$$\left.\begin{aligned}\Sigma\left(m\frac{xdy-ydx}{dt}\right)&=\mathrm{L}\\ \Sigma\left(m\frac{xdz-zdx}{dt}\right)&=\mathrm{M}\\ \Sigma\left(m\frac{ydz-zdy}{dt}\right)&=\mathrm{N}\end{aligned}\right\}\ldots\ldots\ (351\,a).$$

The equations (351 *a*) are independent of the co-ordinates of the centre of gravity, and would undergo no change if forces were applied at that point sufficient to destroy its motion of translation, since such forces would not enter into the expressions L, M, and N; thus, the motion of rotation about the centre of gravity, determined by these equations, is precisely similar to that which would take place if the centre of gravity were immoveable.

Hence we conclude, that *when any body is acted upon by incessant forces applied to its several particles, the body will receive two motions: one of translation, in virtue of which its centre of gravity will be transported in space as though the forces were applied directly to that point; and a second, of rotation about the centre of gravity, as though that point were absolutely at rest.*

General Equations of the Motions of a System of Bodies.

616. Let l, l', l'', &c. represent the velocities lost or gained by the several material points which compose a system, in consequence of the mutual connexions of its parts; the corresponding quantities of motion lost or gained will be ml, $m'l'$, $m''l''$, &c., and, by the principle of D'Alembert, these quantities of motion, when impressed upon the particles m, m', m'', &c. are such as will produce an equilibrium: hence, they must fulfil the conditions of equilibrium expressed in equations (66) and (67).

The components of these quantities of motion, or the forces capable of producing them, estimated in the directions of three rectangular axes, will be

$ml\cos\alpha$, $ml\cos\beta$, $ml\cos\gamma$ ..... components of ml.
$m'l'\cos\alpha'$, $m'l'\cos\beta'$, $m'l'\cos\gamma'$ ..... components of $m'l'$.
$m''l''\cos\alpha''$, $m''l''\cos\beta''$, $m''l''\cos\gamma''$ ..... components of $m''l''$.
&c. &c. &c. &c.

We shall therefore have for the equations of equilibrium,

$$\left.\begin{aligned}\Sigma(ml\,.\cos\alpha)=0\\ \Sigma(ml\,.\cos\beta)=0\\ \Sigma(ml\,.\cos\gamma)=0\end{aligned}\right\}\;\ldots\ldots\;(352).$$

$$\left.\begin{array}{l}\Sigma[ml(x\cos\beta-y\cos\alpha)]=0\\ \Sigma[ml(z\cos\alpha-x\cos\gamma)]=0\\ \Sigma[ml(y\cos\gamma-z\cos\beta)]=0\end{array}\right\}\ldots\ldots(353).$$

617. If the system is retained by a fixed point, the three equations (352) cease to be necessary; the equations (353) being alone sufficient, provided the origin be placed at the fixed point.

618. When there are two fixed points within the system, we connect them by a right line, and assume this line as one of the co-ordinate axes, z for example; the first of equations (353) will then be sufficient to ensure the equilibrium (Arts. 132 and 133).

619. The velocities lost or gained are here indicated by the letters l, l', l'', &c.; but to express these quantities in functions of the incessant forces which solicit the several material points, we shall first consider the particle m, and suppose that the forces acting upon this point have been reduced to three, X, Y, and Z, respectively parallel to the co-ordinate axes. The velocity of the particle m, parallel to the axis of x, at the expiration of the time t, will be expressed by $\frac{dx}{dt}$ (Art. 430); and at the end of the time $t+dt$, this velocity will become $\frac{dx}{dt}+d\frac{dx}{dt}$; this will be the expression for the effective velocity of the particle m.

But if the particle m were perfectly free, the incessant force X would communicate to it in the time dt, a velocity represented by Xdt (Art. 391), and the velocity of m at the expiration of the time $t+dt$, would be expressed by $\frac{dx}{dt}+\mathrm{X}dt$; hence, the velocity lost or gained by the particle m will be equal to

$$\frac{dx}{dt}+\mathrm{X}dt-\left(\frac{dx}{dt}+d\frac{dx}{dt}\right);$$

and by reduction, we shall find that $\mathrm{X}dt-d\frac{dx}{dt}$ will express the velocity lost or gained by the particle m, in the direction of the axis of x. This velocity being multiplied by the mass m, gives

$$m\left(\mathrm{X}dt-d\frac{dx}{dt}\right),$$

for the quantity of motion lost or gained by m, in the direction of the axis of x: we shall therefore have

$$ml\,.\cos\alpha=m\left(\mathrm{X}dt-\frac{d^2x}{dt}\right)\;\ldots\ldots\;(354).$$

In like manner, by considering the velocities lost by m, in directions parallel to the axis of y and z, we shall find

$$ml\,.\cos\beta=m\left(\mathrm{Y}dt-\frac{d^2y}{dt}\right)\;\ldots\ldots\;(355).$$

$$ml\,.\cos\gamma=m\left(\mathrm{Z}dt-\frac{d^2z}{dt}\right)\;\ldots\ldots\;(356).$$

Similar expressions may be obtained for the quantities of motion lost or gained by the particles m', m'', &c.; and by including their sums under the sign Σ, the equations (352) and (353) may be reduced to

$$\left.\begin{aligned}\Sigma\left(m\frac{d^2x}{dt^2}\right)&=\Sigma(m\mathrm{X})\\ \Sigma\left(m\frac{d^2y}{dt^2}\right)&=\Sigma(m\mathrm{Y})\\ \Sigma\left(m\frac{d^2z}{dt^2}\right)&=\Sigma(m\mathrm{Z})\end{aligned}\right\}\;\ldots\ldots\;(357).$$

$$\left.\begin{aligned}\frac{\Sigma[m(xd^2y-yd^2x)]}{dt^2}&=\Sigma[m(\mathrm{Y}x-\mathrm{X}y)]\\ \frac{\Sigma[m(zd^2x-xd^2z)]}{dt^2}&=\Sigma[m(\mathrm{X}z-\mathrm{Z}x)]\\ \frac{\Sigma[m(yd^2z-zd^2y)]}{dt^2}&=\Sigma[m(\mathrm{Z}y-\mathrm{Y}z)]\end{aligned}\right\}\;\ldots\ldots\;(358).$$

Such are the most general forms of the equations expressing the circumstances of motion of a system.

620. The expressions $\mathrm{Y}x-\mathrm{X}y$, $\mathrm{X}z-\mathrm{Z}x$, $\mathrm{Z}y-\mathrm{Y}z$, &c. become equal to zero under the following circumstances: 1°. when the incessant forces acting on the particles m, m', m'', &c. are equal to zero; 2°. when all the forces are directed towards the origin of co-ordinates: 3°. when the forces are such as arise from the mutual attractions of the different parts of the system.

In the first case, the incessant forces being equal to zero, their components must likewise be equal to zero; and hence.

$$X=0,\quad Y=0,\quad Z=0,\quad X'=0,\ \&c.:$$

the second members of equations (358) will therefore disappear.

621. The second members will likewise disappear, when the forces are directed towards the origin of co-ordinates. For, it has been shown (Art. 436), that when the fixed point towards which the forces are directed does not coincide with the origin of co-ordinates, if we represent by a, b, and c the co-ordinates of this point, and by p, p', p'', &c. the distances of the several particles from the fixed point, the components of the forces P, P′, P″, &c., in the directions of the co-ordinate axes, will be expressed by

$$\begin{array}{llll}
P\dfrac{x-a}{p}, & P'\dfrac{x'-a}{p'}, & P''\dfrac{x''-a}{p'}, & \&c., \\[2ex]
P\dfrac{y-b}{p}, & P'\dfrac{y'-b}{p'}, & P''\dfrac{y''-b}{p''}, & \&c., \\[2ex]
P\dfrac{z-c}{p}, & P'\dfrac{z'-c}{p'}, & P''\dfrac{z''-c}{p''}, & \&c.;
\end{array}$$

but, by hypothesis, the origin coincides with the fixed point towards which the forces are directed, and we therefore have

$$a=0,\quad b=0,\quad c=0:$$

hence, the preceding expressions are reduced to

$$\begin{array}{llll}
\dfrac{Px}{p}, & \dfrac{P'x'}{p'}, & \dfrac{P''x''}{p''}, & \&c., \\[2ex]
\dfrac{Py}{p}, & \dfrac{P'y'}{p'}, & \dfrac{P''y''}{p''}, & \&c., \\[2ex]
\dfrac{Pz}{p}, & \dfrac{P'z'}{p'}, & \dfrac{P''z''}{p''}, & \&c.
\end{array}$$

And by substituting these values of the components for X, X′, X″, Y, Y′, Y″, Z, Z′, Z″, &c. in the expressions

$$Yx-Xy,\quad Xz-Zx,\quad Zy-Yz,\quad Y'x'-X'y',\ \&c. \ldots\ldots (359),$$

we shall find each of these expressions equal to zero. Consequently, when the incessant forces which act upon the several particles are constantly directed towards the origin, the expressions (359) become equal to zero, and the second members of equations (358) will therefore disappear.

622. The same consequences may be deduced when the material particles are subjected only to their mutual attractions. For, by putting the second members of the equations (358) under the following forms:

$$\left.\begin{array}{l} m(\mathrm{Y}x-\mathrm{X}y)+m'(\mathrm{Y}'x'-\mathrm{X}'y')+\&c. \\ m(\mathrm{X}z-\mathrm{Z}x)+m'(\mathrm{X}'z'-\mathrm{Z}'x')+\&c. \\ m(\mathrm{Z}y-\mathrm{Y}z)+m'(\mathrm{Z}'y'-\mathrm{Y}'z')+\&c. \end{array}\right\} \ldots\ldots (360),$$

and considering the material points two by two, it is evident that the moving force exerted by the point m upon m' is equal to that exerted by m' upon m. Hence, if X, Y, Z, X′, Y′, Z′, &c. represent the components of the incessant forces P, P′, P″, &c., we shall have

$$m'\mathrm{X}'=-m\mathrm{X}, \quad m'\mathrm{Y}'=-m\mathrm{Y}, \quad m'\mathrm{Z}'=-m\mathrm{Z}, \&c.:$$

eliminating X′ and Y′ by means of these values, the first of the expressions (360) becomes

$$m\mathrm{Y}(x-x')-m\mathrm{X}(y-y') \ldots\ldots (361):$$

but the force whose components are X, Y, and Z being denoted by P, and the distance between the points m and m' by p, the cosines of the angles formed by the direction of the force P with the co-ordinate axes, will be represented respectively, by

$$\frac{x-x'}{p}, \quad \frac{y-y'}{p}, \quad \frac{z-z'}{p};$$

and we shall have

$$\mathrm{X}=\mathrm{P}\frac{x-x'}{p}, \quad \mathrm{Y}=\mathrm{P}\frac{y-y'}{p}, \quad \mathrm{Z}=\mathrm{P}\frac{z-z'}{p}.$$

Substituting these values in the expressions (361), we obtain

$$m\mathrm{P}\,.\,\frac{y-y'}{p}(x-x')-m\mathrm{P}\,.\,\frac{x-x'}{p}(y-y');$$

a quantity evidently equal to zero.

In like manner, it may be proved that the other terms of the expressions (360) destroy each other; it therefore follows, that when the material particles m, m', m'', &c. are subjected only to their mutual attractions, the second members of the equations (358) will disappear; and since this result is independent of the position of the origin, that point may be selected arbitrarily.

623. When either of the three cases just considered presents itself, the equations (358) will reduce to

$$\frac{\Sigma[m(xd^2y-yd^2x)]}{dt^2}=0,$$

$$\frac{\Sigma[m(zd^2x-xd^2z)]}{dt^2}=0,$$

$$\frac{\Sigma[m(yd^2z-zd^2y)]}{dt^2}=0.$$

The quantities included within the brackets being exact differentials, these equations may be written under the form

$$\frac{\Sigma[m\,.\,d(xdy-ydx)]}{dt^2}=0,$$

$$\frac{\Sigma[m\,.\,d(zdx-xdz)]}{dt^2}=0,$$

$$\frac{\Sigma[m\,.\,d(ydz-zdy)]}{dt^2}=0.$$

And by multiplying by dt, and integrating with respect to the time, denoting the arbitrary constants by a, a', and a'', we shall have

$$\left.\begin{array}{l}\Sigma[m(xdy-ydx)]=adt\\ \Sigma[m(zdx-xdz)]=a'dt\\ \Sigma[m(ydz-zdy)]=a''dt\end{array}\right\}\ldots\ldots (362).$$

624. To understand the signification of these integrals, draw the three rectangular axes Ax, Ay, and Az (*Fig.* 217), and call AP$=x$, PQ$=y$: let AQ, the projection of the radius vector Am on the plane of x, y, be denoted by r, and the angle formed by AQ with the axis of x by θ; the infinitely small arc QQ$'$ described with the radius r will be expressed by $rd\theta$; the right-angled triangle APQ gives

$$x=r\,.\cos\theta,\quad y=r\,.\sin\theta\,;$$

and, by differentiating, we obtain

$$dx=-r\,.\sin\theta\,.\,d\theta+\cos\theta\,.\,dr,$$
$$dy=r\,.\cos\theta\,.\,d\theta+\sin\theta\,.\,dr.$$

Substituting these values in the expression $xdy-ydx$, we find

$$xdy-ydx=r^2d\theta=2\times\tfrac{1}{2}r\times rd\theta=2\,.\,\text{area QAQ}'\,;$$

and therefore,

$$m(xdy-ydx)=2m(\text{area } QAQ').$$

By forming similar products for the other masses m', m'', &c., we shall find that the quantity $\Sigma[m(xdy-ydx)]$ is composed of the sum of the products formed by multiplying each mass m, m', m'', &c. by twice the area of the elementary surface described by the projection of its radius vector Am on the plane of x, y, in the time dt.

625. If we integrate again with respect to the time, the equations (362) will give

$$\left.\begin{aligned}&\int\Sigma[m(xdy-ydx)]=at+b\\&\int\Sigma[m(zdx-xdz)]=a't+b'\\&\int\Sigma[m(ydz-zdy)]=a''t+b''\end{aligned}\right\}\dots\dots(363);$$

and if the areas described be supposed to commence from the instant when $t=0$, the constants b, b', and b'' will be equal to zero, and the preceding equations will reduce to

$$\int\Sigma[m(xdy-ydx)]=at,$$
$$\int\Sigma[m(zdx-xdz)]=a't,$$
$$\int\Sigma[m(ydz-zdy)]=a''t.$$

These equations express that *the sums of the products formed by multiplying each mass by the projection of the area described by its radius vector, are constantly proportional to the times employed in describing these areas.*

This enunciation contains *the principle of the preservation of areas* in its most general form.

626. The system here considered has been supposed free; but if it were retained by a fixed point, the equations (358) would only be applicable when the origin was taken at this point: the same may be said of equations (363), which result from (358). Thus the principle of areas then becomes less general, the origin being no longer arbitrary.

627. It has been shown (Arts. 132 and 133) that when the system contains two fixed points, it will be necessary to satisfy but one of the general equations of equilibrium (67). The same is true with respect to equations (358); and therefore but one of the equations (362) will be satisfied: thus, the principle of areas is only true in this case with respect to one of the co-ordinate planes.

628. By comparing the results obtained in Art. 155 with

those of Art. 153, we shall find that the quantities A, B, and C represent, in Art. 155, the sums of the moments of the projections of the forces on the co-ordinate planes, these moments being taken with reference to the origin. Thus these sums will be the same as those denoted by a, a', a'' in equations (362). Hence, the sum of the projections on the principal plane given by equation (79), will, in the present instance, be expressed by

$$\sqrt{\left(\frac{(\Sigma[m(xdy-ydx)])^2}{dt^2}+\frac{(\Sigma[m(zdx-xdz)])^2}{dt^2}+\frac{(\Sigma[m(ydz-zdy)])^2}{dt^2}\right)}.$$

This expression may be simplified by putting it under the form

$$\sqrt{(a^2+a'^2+a''^2)};$$

and replacing the functions A, B, and C in equations (81) by their values a, a', and a'', we obtain the following expressions for the cosines of the angles formed by the principal plane with the co-ordinate planes:

$$\cos\alpha=\frac{a}{\sqrt{(a^2+a'^2+a''^2)}},\quad \cos\beta=\frac{a'}{\sqrt{(a^2+a'^2+a''^2)}},$$

$$\cos\gamma=\frac{a''}{\sqrt{(a^2+a'^2+a''^2)}}.$$

The angles α, β, γ are constant; and hence we conclude that the position of the principal plane remains invariable during the motions of the several particles of which the system is composed.

General Principle of the Preservation of the Motion of the Centre of Gravity.

629. In discussing the circumstances of motion of a system of material particles, acted upon by incessant forces, it was proved that the centre of gravity of the entire system has the same motion as though the several forces were applied directly to that point. Thus, denoting by $x_{,}$, $y_{,}$ and $z_{,}$ the variable co-ordinates of the centre of gravity, we shall have, as in Art. 614,

$$MX_{,}=\Sigma(mX),\quad MY_{,}=\Sigma(mY),\quad MZ_{,}=\Sigma(mZ)\;\ldots\ldots\;(366).$$

and,

$$\frac{d^2x_{,}}{dt^2}=X_{,,}\quad \frac{d^2y_{,}}{dt^2}=Y_{,,}\quad \frac{d^2z_{,}}{dt^2}=Z_{,}\ldots\ldots (367).$$

630. If the material points which compose the system be subjected only to the action of forces arising from their mutual attractions, the equations (367) will reduce to

$$\frac{d^2x_{,}}{dt^2}=0,\quad \frac{d^2y_{,}}{dt^2}=0,\quad \frac{d^2z_{,}}{dt^2}=0\,;$$

these equations being integrated give

$$\frac{dx_{,}}{dt}=a,\quad \frac{dy_{,}}{dt}=b,\quad \frac{dz_{,}}{dt}=c,$$

and by a second integration we find

$$x_{,}=at+a',\quad y_{,}=bt+b',\quad z_{,}=ct+c_{,}:$$

eliminating t, we have

$$x_{,}-a'=\frac{a}{c}(z_{,}-c'),\quad y_{,}-b'=\frac{b}{c}(z_{,}-c').$$

These equations appertain to a right line in space, and the motion of the centre of gravity will therefore be rectilinear. This motion will also be uniform; for we have the velocity of the centre of gravity expressed by

$$\sqrt{\left(\frac{dx_{,}^2}{dt^2}+\frac{dy_{,}^2}{dt^2}+\frac{dz_{,}^2}{dt^2}\right)}=\sqrt{(a^2+b^2+c^2)},$$

which is evidently a constant quantity.

631. If the masses m, m', m'', &c. be subjected to the action of constant forces whose directions are parallel to a given line, we may adopt this line as one of the co-ordinate axes, z for example, and the equations expressing the circumstances of motion of the centre of gravity, then become

$$\frac{d^2x_{,}}{dt^2}=0,\quad \frac{d^2y_{,}}{dt^2}=0,\quad \frac{d^2z_{,}}{dt^2}=Z\,;$$

and it may then be proved, as in Arts. 518 and 519, that the trajectory described by the centre of gravity is a parabola.

632. Finally, it may be shown that if two or more of the bodies composing the system impinge against each other during the motion, the velocity of the centre of gravity will remain unchanged. For, by the nature of the centre of gravity, we have

$$\mathrm{M}x_{,}=\Sigma(mx),\quad \mathrm{M}y_{,}=\Sigma(my),\quad \mathrm{M}z_{,}=\Sigma(mz):$$

differentiating with respect to the time t, we obtain

$$\mathrm{M}\frac{dx_{,}}{dt}=\Sigma\left(m\frac{dx}{dt}\right),\quad \mathrm{M}\frac{dy_{,}}{dt}=\Sigma\left(m\frac{dy}{dt}\right),\quad \mathrm{M}\frac{dz_{,}}{dt}=\Sigma\left(m\frac{dz}{dt}\right).$$

And if we denote by a, a', a'', &c. the velocities of the particles m, m', m'', &c. before the collision, and by A, A′, A″, &c. the corresponding velocities after collision, these values, substituted in the first of the preceding equations will give

$$\Sigma(ma)=\text{ the value of }\mathrm{M}\frac{dx_{,}}{dt}\text{ before collision,}$$

$$\Sigma(m\mathrm{A})=\text{ the value of }\mathrm{M}\frac{dx_{,}}{dt}\text{ after collision.}$$

Thus the sum of the quantities of motion lost by the impact, in the direction of the axis of x, will be $\Sigma(ma)-\Sigma(m\mathrm{A})$. In like manner, the sums of the quantities of motion lost in the direction of the axes of y and z respectively, will be

$$\Sigma(mb)-\Sigma(m\mathrm{B}),\quad\text{and}\quad \Sigma(mc)-\Sigma(m\mathrm{C}):$$

but, by the principle of D'Alembert, these quantities should maintain the system in equilibrio; and we therefore have

$$\Sigma(ma)-\Sigma(m\mathrm{A})=0,\quad \Sigma(mb)-\Sigma(m\mathrm{B})=0,\quad \Sigma(mc)-\Sigma(m\mathrm{C})=0;$$

hence, the expressions $\frac{dx_{,}}{dt}$, $\frac{dy_{,}}{dt}$, $\frac{dz_{,}}{dt}$, which represent the velocities of the centre of gravity parallel to the co-ordinate axes, remain unchanged by the act of collision.

633. This property of the centre of gravity, in virtue of which its motion is independent of the mutual actions of the parts of the system, constitutes *the principle of the preservation of the motion of the centre of gravity.*

PART THIRD.

HYDROSTATICS.

OF THE PRESSURE OF FLUIDS.

634. A fluid is a collection of material particles, which yield to the slightest effort, and which move freely among each other in all directions.

When the material particles adhere to each other in any degree, the fluid is said to be imperfect; in the following pages the particles will be supposed entirely destitute of any adhesion.

635. Fluids are divided into incompressible and compressible or elastic fluids.

Incompressible fluids are such as always occupy the same volume at the same temperature; such are water, mercury, wine, &c.

Elastic fluids are those whose volumes admit of change by the application of pressure; such are the vapour of water, atmospheric air, and the different gases.

636. Let ABCD (*Fig.* 218) represent a vessel entirely closed, and filled with a fluid destitute of weight: if two apertures EF and HI, having equal surfaces, be pierced in this vessel, and if pistons K and L be applied to these apertures, and urged by forces RK and SL, equal in intensity, and directed perpendicularly to the surfaces HI and EF, these forces will remain in equilibrio. Hence, it is necessary that the pressure exerted upon the surface EF should be communicated to the surface HI, through the intervention of the fluid medium; and this can only happen provided the particles of the fluid experience the same pressure at every point of the fluid mass. Adopting the result of this experiment as a basis, we can establish the following principle:

The characteristic property of fluids is that they transmit a pressure applied to them, equally in all directions.

637. To express analytically this property, which is termed the *principle of equal pressure*, we shall consider a fluid mass enclosed in a vessel AL (*Fig.* 219) having the form of a rectangular parallelopiped, the base of which is horizontal. Let a piston be applied to the upper surface EH of the fluid, and let it be urged downward by a force P, acting in the vertical direction: the base of the vessel will experience the same pressure as though the force were applied directly to it; and each portion of the base will support a pressure proportional to its extent; so that if A denote the area ABCD, and a the area $Abcd$, of a portion of this base; and if p denote the pressure sustained by a, the value of p will result from the following proportion,

$$\mathrm{A} : a :: \mathrm{P} : p.$$

Let a represent the unit of surface; we shall then have

$$p = \frac{\mathrm{P}}{\mathrm{A}};$$

hence, if ω represent the ratio between the surface $Ab'c'd'$, and the surface $Abcd$ assumed as the unit, the pressure P′ supported by the surface $Ab'c'd'$, will be expressed by

$$\mathrm{P}' = p\omega \; \ldots\ldots \; (381);$$

and since all portions of the fluid mass must sustain equal pressures for the same extent of surface, it follows that if the surface containing ω units were situated in any other portion of the vessel, on the sides for example, it would still sustain the same pressure $p\omega$.

638. When the surface pressed is indefinitely small, it may be represented by the elementary rectangle $dxdy$; and the pressure exerted by the piston on this elementary portion of the surface of the vessel, will be expressed by $pdxdy$: this expression will be equally applicable in whatever portion of the vessel the element may be situated, and whether the surface be plane or curved.

639. In the preceding paragraphs, the fluid has been supposed subjected merely to the action of the pressure applied at its surface; but when the particles of the fluid are acted

upon by incessant forces, the pressure will cease to be constant throughout the mass. In this case, the pressure sustained by the fluid arises from two distinct causes: 1°. a pressure resulting from the force P applied to the surface, and equally distributed throughout the mass; and, 2°. the pressure arising from the action of the incessant forces. The latter pressure is usually different in different parts of the fluid mass, since each particle may be acted on by a force having any intensity.

640. To offer an example of this second kind of pressure, let the fluid contained in the vessel ABCD (*Fig.* 218) be considered heavy: then we must regard each particle as acted on by the force of gravity.

We shall find in the sequel, in discussing the properties of heavy fluids, that the principle of equal pressure is greatly modified by this circumstance. It follows from the preceding remarks, that the pressure p should in general be regarded as variable, in passing from one point to another of a fluid mass, when the particles are acted upon by incessant forces. In this case, the pressure p exerted at any point whose co-ordinates are x, y, z, when referred to the unit of surface, must be understood to denote the pressure which would be exerted upon a unit of surface, if every point in this unit should sustain a pressure equal to that exerted at the point x, y, z.

General Equations of the Equilibrium of Fluids.

641. Let a fluid particle solicited by incessant forces be supposed to rest in equilibrio in a fluid mass, and let it be required to determine the equations necessary to establish the state of equilibrium.

For this purpose, let the co-ordinate plane of x, y be assumed horizontal and above the fluid mass, which we will suppose divided into infinitely small rectangular parallelopipeds by planes parallel to the co-ordinate planes. Let dM represent one of these elements whose co-ordinates are x, y, and z: the volume of this element will be expressed by $dxdydz$; and by multiplying this volume by the density D, supposed constant throughout the element, we shall have $D.dxdydz$

for the expression of the elementary mass of the fluid: hence, we derive the equation

$$dM = D \, . \, dxdydz \; \ldots\ldots \; (382).$$

If X, Y, and Z represent the incessant forces which act upon the element dM, and which are supposed constant throughout the extent of this element, XdM, YdM, and ZdM will express the moving forces exerted upon the elementary parallelopiped, and these forces, acting conjointly with the pressure sustained by the several faces of the element, should maintain this element in equilibrio. Let the superior surface $dxdy$ of the parallelopiped be extended (*Fig.* 220) until its area becomes equal to the assumed unit represented by BC; and let the pressure p sustained throughout this unit be conceived constant, and equal to that exerted at each point of the face $dxdy$. When the ordinate $BD = z$ is changed into $DE = z + dz$, the pressure p, which varies with z, will become

$$p + \frac{dp}{dz}dz,$$

and will express the pressure exerted on the unit of surface, each point of which sustains a pressure equal to that supported by the points in the base EF of the parallelopiped. Consequently, to obtain the total pressures on the superior base BG and on the inferior base EF of the element, we must multiply the surfaces BG and EF, each of which is equal to $dxdy$, by the respective pressures exerted upon their unit of surface: thus, we shall obtain for the pressures supported by BG and EF,

$$pdxdy, \quad \text{and} \quad \left(p + \frac{dp}{dz}dz\right)dxdy;$$

the first of these pressures is exerted downwards, and the latter upwards. Their difference will be a pressure exerted upwards, if we suppose the pressure to increase with the co-ordinate z, and it will be expressed by

$$\frac{dp}{dz}dzdxdy:$$

and since this difference should sustain in equilibrio the vertical force ZdM, we shall have

$$\frac{dp}{dz}dzdxdy=ZdM:$$

substituting for dM its value given by equation (382), and reducing, we find

$$\frac{dp}{dz}=DZ.$$

In like manner, by denoting the lateral pressures on a unit of surface exerted against the faces $dxdz$, $dydz$, by q and r, we shall obtain

$$\frac{dq}{dy}=DY, \quad \frac{dr}{dx}=DX.$$

It has been shown (Art. 640) that the pressures exerted upon any one of the faces is composed of the pressure uniformly distributed throughout the fluid, and of the pressure due to the incessant forces. Thus, to estimate the pressure $qdxdz$, exerted upon the face $dxdz$, it is obvious that this pressure may be considered as resulting from, 1°. The pressure exerted upon the superior base, which is transmitted equally throughout the parallelopiped; and, 2°. The pressure due to the incessant forces exerted upon the particles which compose the parallelopiped. But the pressure exerted upon the upper base being $pdxdy$, it will be transmitted to the face $dxdz$, exerting a pressure $pdxdz$ proportional to the area of this face.

The incessant forces being XdM, YdM, and ZdM respectively, the pressure arising from their joint action will be a function of their intensities, which we shall represent by

$$F(XdM, YdM, ZdM);$$

and we shall thus obtain

$$qdxdz=pdxdz+F(Xdm, Ydm, Zdm) \ldots\ldots (383).$$

The function represented by

$$F(Xdm, Ydm, Zdm)$$

must be such that it will disappear when the forces are supposed equal to zero: hence it is necessary that every term of the function should contain at least one of the factors XdM, YdM, or ZdM: and by arranging the terms with reference to the powers of dM, commencing with the least, we may suppose

$$F(XdM, YdM, ZdM)=LXdM+NYdM+PZdM+\&c.$$

Substituting this value in equation (383), we shall have

$$qdxdz=pdxdz+LXdM+NYdM+PZdM+\&c.;$$

and replacing dM by its equal $Ddxdydz$, this equation will become

$$qdxdz=pdxdz+DLXdxdydz+DNYdxdydz \\ +DPZdxdydz+\&c.:$$

dividing by $dxdz$, there results

$$q=p+DLXdy+DNYdy+DPZdy+\&c. \ldots\ldots (384).$$

The terms $DLXdy$, $DNYdy$, $DPZdy$, &c. being infinitely small with respect to p, it follows that the equation (384) may be reduced to

$$q=p.$$

In like manner, it may be demonstrated that $r=p$; and hence, the equations of equilibrium will become

$$\frac{dp}{dz}=DZ, \quad \frac{dp}{dy}=DY, \quad \frac{dp}{dx}=DX \ldots\ldots (385).$$

If we multiply these equations by dz, dy, and dx respectively, and take their sum, we shall obtain an expression for the differential of the pressure, when the co-ordinates x, y, and z are supposed to vary together; thus,

$$dp=\frac{dp}{dx}dx+\frac{dp}{dy}dy+\frac{dp}{dz}dz=D(Xdx+Ydy+Zdz) \ldots\ldots (386).$$

Such is the equation which, when integrated, will determine the pressure upon the unit of surface at any point of the fluid.

Application of the General Equations of Equilibrium to Incompressible Fluids.

642. Let us suppose an incompressible homogeneous fluid to be in equilibrio in a vessel capable of opposing an indefinite resistance to pressure: the pressure p exerted upon the unit of surface, at a point whose co-ordinates are $x=a$, $y=b$, $z=c$, will be determined by substituting the values a, b, and c for x, y, and z, in the integral of equation (386): and if the density D be supposed constant, the determination of the

value of p will depend on the possibility of integrating the formula

$$Xdx+Ydy+Zdz \dots\dots (387).$$

This integration will always be possible, when the preceding expression is an exact differential of the variables x, y, and z.

Let it be supposed that this condition is fulfilled, and that the pressure at any point on the sides or bottom of the vessel has been determined; this pressure will be destroyed by the resistance of the vessel. But if we consider a point in the free surface of the fluid, and suppose that no exterior pressure is applied to the fluid by means of a piston or otherwise, it is obvious that the pressure at such point will be equal to zero. The same being true for every point in the free surface of the fluid, it follows that in passing from any point in the surface of the fluid to a consecutive point in the same surface, the pressure p will remain invariable, being equal to zero at each of these points; hence $dp=0$, and the equation (386) condered as applicable to points situated in this surface, will reduce to

$$Xdx+Ydy+Zdz=0 \dots\dots (388).$$

This equation will likewise appertain to the surface of the fluid when this surface experiences a constant pressure, that of the atmosphere for example, since we shall still have $dp=0$. It will also subsist for those points within the fluid mass which sustain equal pressures.

643. When the expression (387) is an exact differential, and the equation (388) is satisfied, we shall have $dp=0$, and the pressure, if it exist, must be constant. But, in this case, in order that the equilibrium may be preserved, it is necessary that the resultant of the forces exerted upon each particle in the surface, and directed towards the interior of the fluid, should be normal to the surface of the fluid: for if it were not, we might decompose this resultant into two forces, one normal and the other tangent to the surface; and it is evident that the latter would impart a motion to the fluid particle.

644. This condition is likewise indicated by the equation

(388); for, let x', y', and z' represent the co-ordinates of a particle in the surface of the fluid, and X, Y, and Z the incessant forces applied to this particle. The general equations of the normal to a curved surface at the point x', y', z', are

$$\left.\begin{array}{l} x-x'=-\frac{dz'}{dx'}(z-z') \\ y-y'=-\frac{dz'}{dy'}(z-z') \end{array}\right\} \ldots\ldots (389);$$

and if we substitute in these equations the values of $\frac{dz'}{dx'}$ and $\frac{dz'}{dy'}$ determined by equation (388), the equations (389) will become those of the normal to the surface of which (388) is the equation. But by regarding X, Y, and Z as functions of the co-ordinates x, y, and z, and employing the usual notation, the equation (388) will give

$$-\frac{dz'}{dx'}=\frac{X}{Z}, \quad -\frac{dz'}{dy'}=\frac{Y}{Z}.$$

Substituting these values in (389), we find, for the equations of the normal at the point x', y', z',

$$x-x'=\frac{X}{Z}(z-z'), \quad y-y'=\frac{Y}{Z}(z-z').$$

These equations are precisely similar to those of the resultant of the forces X, Y, and Z, found in Art 57.

645. The equation (388), when susceptible of being integrated, leads to several remarkable consequences. For, if we represent the integral of this equation by $F(x, y, z)+C$, and make $C=-A$, we shall have

$$F(x, y, z)=A.$$

If we assign to A arbitrary values successively increasing, such as 0, a, a', a'', &c., we shall obtain the equations

$$\begin{array}{c} F(x, y, z)=0, \\ F(x, y, z)=a, \\ F(x, y, z)=a', \\ F(x, y, z)=a'', \\ \ldots\ldots\ldots\ldots \\ F(x, y, z)=a^{(n)}, \\ \&c. \quad \&c. \end{array}$$

Each of these equations being differentiated will produce equation (388), and among them will be found that appertaining to the surface of the fluid, which is supposed to have produced equation (388) by differentiation.

Let this equation be represented by $F(x, y, z) = a^{(n)}$: then the other equations will appertain to different surfaces, each of which will possess the property, that the resultant R of the forces X, Y, Z, exerted upon any particle situated in such surface, will be perpendicular to the surface.

The directions of the forces being cut perpendicularly by the surfaces of constant pressure, such surfaces are said to be *level*. If we suppose the arbitrary constants 0, a, a', a'', &c. to differ by indefinitely small increments, the fluid mass will be divided by these level surfaces into a series of extremely thin layers, which are denominated *level strata*.

646. It follows, from the preceding remarks, that when the particles of the fluid are solicited by forces constantly directed towards a fixed point, its exterior will assume the spherical form. The same consequence may be deduced analytically. For, let the origin be taken at the centre of attraction, and denote by x, y, z the co-ordinates of a particle dM in the surface of the fluid: the distance of the point x, y, z from the origin will be expressed by $\sqrt{(x^2 + y^2 + z^2)}$. If this distance be denoted by r, and the force of attraction exerted upon the particle dM by λ, the cosines of the angles formed by the direction of this force with the co-ordinate axes will be expressed by $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$; and the components of the force λ will be

$$X = -\lambda\frac{x}{r}, \quad Y = -\lambda\frac{y}{r}, \quad Z = -\lambda\frac{z}{r};$$

the negative signs are prefixed to these components because they tend to diminish the co-ordinates of the particle dM. By substituting these values in equation (388), we shall obtain for the differential equation of the surface of the fluid

$$\frac{\lambda}{r}(xdx + ydy + zdz) = 0 \; \ldots\ldots \; (390).$$

Suppressing the common factor $\frac{\lambda}{r}$, and integrating, we find

$$x^2+y^2+z^2=C,$$

an equation appertaining to a spherical surface; hence the surface of the fluid will be spherical.

647. If the radius of the sphere be very great in comparison with the extent of the surface, as is the case when we consider a small portion of the earth's surface, the curvature will be insensible, and the surface may therefore be regarded as a plane.

648. The integration of equation (390) was effected immediately in consequence of equation (388) becoming, in that example, a particular case of the theorem demonstrated in Art. 436, relative to forces directed to fixed centres. It is by virtue of this theorem that equation (388) will always be integrable in such cases as refer to the equilibrium of fluids resting upon fixed surfaces.

649. If, in equation (386), we replace the quantity within the brackets by its equal $d[F(x, y, z)]$, we shall obtain

$$dp=D\times d[F(x, y, z)];$$

or, by division,

$$\frac{dp}{D}=d[F(x, y, z)] \ldots\ldots (391).$$

But $d[F(x, y, z)]$ being by hypothesis an exact differential, $\frac{dp}{D}$ must likewise be an exact differential; hence, D will contain no variable except p; this condition may be expressed by the equation

$$D=fp \ldots\ldots (392).$$

If the pressure p be supposed constant, the density D will be likewise constant, and (391) will reduce to

$$d[F(x, y, z)]=0.$$

The integration of this equation will reproduce that already found in Art. 645, the properties of which have been discussed.

650. The fluid being still supposed incompressible, but heterogeneous, the density D will be variable; and in order that the pressure p may be determinate, the quantity $D(Xdx+Ydy+Zdz)$ must be an exact differential: but if

$Xdx+Ydy+Zdz$ be likewise supposed an exact differential, it will appear, as in equation (392), that the density will be always a function of the pressure. Thus the pressure and density will become constant together, and will remain invariable for all points situated in a level stratum.

We conclude, therefore, that a heterogeneous fluid mass cannot remain in equilibrio, unless it be disposed in such manner that each of the level strata shall be of equal density throughout. The law of variation in the density in passing from one stratum to another, will depend on the manner in which D is expressed in functions of x, y, and z: and since the nature of the function is entirely arbitrary, the law of the density will likewise be arbitrary.

Application of the General Equations of Equilibrium to Elastic Fluids.

651. The characteristic property of an elastic fluid is its power of sustaining compression, and subsequently regaining its original volume and elasticity, when the compressing force is removed.

Thus, a fluid which is elastic exerts in addition to the pressure due to the forces which act upon it, an effort arising from the elasticity of its particles.

It has been ascertained experimentally, that this effort, which is called the *elastic force of the fluid*, is proportional to its density, so long as the temperature remains invariable.

Thus, if we suppose the temperature to remain constant, and represent by P that pressure exerted upon the unit of surface which is necessary to produce a certain density assumed as the unit, this density will be doubled when the pressure becomes 2P; trebled when the pressure becomes 3P, &c.; and hence, if the density be expressed by D, the corresponding pressure will be PD. This pressure being denoted by p, we shall have

$$p=\mathrm{PD} \ldots\ldots (393);$$

the quantity p represents, as heretofore, the pressure exerted upon the unit of surface.

652. By combining equation (393) with the equation

$$dp = D(Xdx + Ydy + Zdz),$$

there results

$$\frac{dp}{p} = \frac{Xdx + Ydy + Zdz}{P} \ldots\ldots (394);$$

and by integration, we have

$$\log p = \int \frac{Xdx + Ydy + Zdz}{P} + C.$$

653. The temperature being supposed constant throughout the mass, and the nature of the fluid particles everywhere the same, the quantity P will be constant, and may therefore be placed without the integral sign: thus, by representing the constant C by log C′, we shall have

$$\log p = \frac{\int(Xdx + Ydy + Zdz)}{P} + \log C';$$

or, if we denote by e the base of the Naperian system, this equation will reduce to

$$\log p = \log e^{\frac{\int(Xdx+Ydy+Zdz)}{P}} + \log C':$$

reducing, and passing from logarithms to numbers, we find

$$p = C'e^{\frac{\int(Xdx+Ydy+Zdz)}{P}}.$$

This value being substituted in equation (393), we obtain

$$D = \frac{C'e^{\frac{\int(Xdx+Ydy+Zdz)}{P}}}{P}.$$

The pressure and density being both functions of the quantity $\int(Xdx + Ydy + Zdz)$, they will become constant at the same time; and hence, the density of the fluid throughout each level stratum will remain invariable. The value of the density in any stratum results immediately from the preceding equation.

654. It should be remarked, that in the case of elastic fluids, the equation

$$Xdx + Ydy + Zdz = 0$$

cannot be deduced from the hypothesis of $p=0$: for, if we suppose $p=0$, the equation will give $D=0$; and hence, we perceive that it would be necessary that the density of the fluid should be likewise equal to zero; a supposition which would destroy the existence of the fluid.

We conclude, therefore, that in an elastic fluid, the pressure cannot be equal to zero at the surface of the fluid, as is the case with incompressible fluids. Thus, a mass of elastic fluid cannot be in equilibrio unless contained in a close vessel, or extended indefinitely in space.

Of the Pressure of Heavy Fluids.

655. It is now proposed to examine the circumstances of equilibrium in fluids whose particles are acted on by the force of gravity. For this purpose, let it be supposed that a vessel is placed upon a horizontal plane, and filled with water, or other heavy fluid, to a certain height. The surface of the fluid, as has been demonstrated, will assume a horizontal position; let this surface be assumed as the plane of x, y, and let the co-ordinates z be reckoned positive downwards; the force of gravity being the only force exerted upon the fluid particles, we shall have

$$X=0, \quad Y=0, \quad Z=g;$$

and the equation (386) will become

$$dp=\mathrm{D}g\,dz.$$

The density of the fluid and the intensity of gravity being supposed constant, the integration of this equation will give

$$p=\mathrm{D}gz+\mathrm{C} \; \ldots\ldots \; (395).$$

To determine the value of the constant C, we make $z=0$, and since the pressure p is equal to zero at the same time, we deduce $\mathrm{C}=0$: thus the equation (395) is reduced to

$$p=\mathrm{D}gz \; \ldots\ldots \; (396).$$

656. If a horizontal plane be drawn below the surface of the fluid, every point in such plane will have a common ordinate z; and the pressure $p=\mathrm{D}gz$ will therefore be constant throughout this plane.

657. Let h represent the distance between the surface of the fluid and the horizontal base of the vessel; the pressure supported by the unit of surface of the base will be determined

by equation (396), in which we replace z by h, and thus obtain

$$p=\mathrm{D}gh \ldots\ldots (397).$$

Let p' represent the pressure supported by the entire base, which is supposed to contain b units of surface: the quantity p will be contained b times in p': we therefore have

$$p'=bp \ldots\ldots (398),$$

and by substituting for p its value given in equation (397), we find

$$p'=\mathrm{D}ghb \ldots\ldots (399).$$

But bh represents the volume of a prism whose base is b, and height h; and by multiplying this volume by the density D, we obtain $bh\mathrm{D}$ for the mass of the prism: therefore $bgh\mathrm{D}$ will express the weight of such prism; and hence, it appears that the base b supports a pressure equal to the weight of the column of fluid which rests immediately upon it.

658. The pressure p', exerted by the same fluid, being dependent only on the base b and height h, it follows that the pressures supported by the bases of different vessels will be equal, whatever may be the forms of the vessels, provided their bases, and the heights of the fluid above them, be respectively equal.

659. To determine the lateral pressure exerted against the sides of the vessel, let $d\omega$ represent the element of this surface, and z the distance of the element from the surface of the fluid; the pressure p (referred to the unit of surface), which is supported by the element $d\omega$, will be determined by equation (396): this value being substituted in equation (399), and the area b being replaced by $d\omega$, we obtain $\mathrm{D}\,.\,gz\,.\,d\omega$ for the expression of the entire pressure on the element $d\omega$. A similar expression may be obtained for the pressure upon each element; and since the pressures will be exerted in parallel directions when the side of the vessel is supposed plane, we shall have, for the total pressure exerted against the side,

$$p'=\int \mathrm{D}gz\,d\omega.$$

The second member of this equation contains two variables, one of which must be eliminated before the integration can be effected. This elimination is readily accomplished when the figure of the surface ω is known.

660. Let it be required, for example, to determine the pressure exerted against the inclined rectangle ACDB (*Fig.* 221), whose sides AB and CD are parallel to the horizon. Denote by b and l the base AB and length BD of the rectangle, and conceive its surface to be divided into an infinite number of elements, by lines parallel to AB or CD; the pressure will be the same upon every point of the same element. Let v denote the distance Df of any one element af from the base CD; the height of this element will be expressed by $dv=ae$, and the surface of the element by

$$ab \times ae = bdv\,;$$

substituting this value for $d\omega$ in the expression $\int \mathrm{D}gzd\omega$, we obtain

$$\int \mathrm{D}gzd\omega = \int \mathrm{D}gzbdv:$$

such will be the expression for the pressure exerted upon the surface ABDC. The integral should be taken between the limits $v=0$ and $v=l$, the variable z being previously eliminated. To effect this elimination, let ϕ represent the angle BDL included between the plane of the rectangle and the vertical line NL, and a the distance DN of the superior base CD from the surface of the fluid; we shall have

$$\mathrm{K}f \text{ or } \mathrm{LN} = \mathrm{ND} + \mathrm{DL}\,;$$

or,

$$z = a + v \,.\, \cos\phi:$$

hence, the pressure exerted upon the surface will be expressed by

$$p' = \int \mathrm{D}g(a + v \,.\, \cos\phi)bdv\,;$$

and by performing the integration indicated, we find

$$p' = \mathrm{D}gb(av + \tfrac{1}{2}v^2 \cos\phi) + \mathrm{C}.$$

The integral being taken between the limits $v=0$, and $v=l$, we obtain

$$p' = \mathrm{D}gb(al + \tfrac{1}{2}l^2 \cos\phi).$$

661. To determine the point of application of the resultant of all the pressures exerted upon the rectangle, we remark, in the first place, that this point must be situated upon the line

EH, which bisects the sides AB and CD. We next regard the pressures exerted upon the different points of the surface ABDC as parallel forces, and determine their moments with reference to a vertical plane passing through the horizontal line CD: the pressure sustained by the element *abfe* being $Dgzbdv$, its moment will be expressed by $Dgzbdv \times v . \sin\phi$; and by denoting the distance EG of the point of application of the resultant from the line CD by $v_{,,}$ the principle of moments will give

$$p'v_{,} \sin\phi = \sin\varphi \int Dgzbvdv;$$

or,

$$p'v_{,} = \int Dgzbvdv.$$

If, in this expression, we replace z by its value determined in the preceding Art., we shall obtain

$$p'v_{,} = Dgb\int(avdv + \cos\varphi . v^2 dv);$$

whence, by integration,

$$p'v_{,} = Dgb\left(\frac{av^2}{2} + \cos\varphi\frac{v^3}{3}\right) + C.$$

The integral being taken between the limits $v=0$ and $v=l$, there results

$$p'v_{,} = Dgb\left(\frac{al^2}{2} + \cos\varphi\frac{l^3}{3}\right);$$

and by substituting for p' its value, we find, after reduction,

$$v_{,} = \frac{\frac{al}{2} + \cos\varphi\frac{l^2}{3}}{a + \cos\varphi\frac{l}{2}}$$

Having found the pressures exerted upon the base and upon each of the sides of the vessel, we combine these pressures, and determine their resultant: such resultant will express the entire pressure produced by the fluid.

662. We will next consider a body immersed in a homogeneous heavy fluid: the pressure exerted by this fluid against any portion of the surface of the body may be determined by the method for finding the pressure against the sides of a vessel; but when it is required to consider the total pressure exerted against the surface of a body immersed

in a fluid, we commonly employ the following theorems, the truth of which will be demonstrated.

1°. *The pressures exerted upon the surface of a body entirely immersed in a fluid have a single resultant, which is vertical and directed upwards.*

2°. *The resultant of all the pressures is equal in intensity to the weight of the fluid displaced.*

3°. *The line of direction of this resultant passes through the centre of gravity of the displaced fluid.*

4°. *The horizontal pressures destroy each other.*

To establish the truth of these propositions, let us suppose a vessel ADE (*Fig.* 222) to be filled with a heavy fluid in equilibrio, and let a portion of this fluid KL be conceived to become solid, its density remaining unchanged: the state of equilibrium will not be disturbed by this change. But this solid is urged downwards by a force equal to its weight, applied at its centre of gravity. This force can only be destroyed by the resultant of all the pressures exerted by the fluid against the solid; hence, it follows that these pressures must have a single resultant equal in intensity to the weight of the displaced fluid, and that this resultant must be applied at the centre of gravity of the displaced fluid, and be directed vertically upwards. Moreover, as the direction of the resultant is vertical, the horizontal pressures will mutually destroy each other.

When a body is partially immersed in a fluid, an equilibrium cannot subsist unless the centres of gravity of the body and of the fluid displaced be situated upon the same vertical line: this condition will necessarily be fulfilled when the body is entirely immersed, provided it be homogeneous; since its centre of gravity will then coincide with that of the fluid displaced.

The buoyant effort exerted by the fluid being directed along a line which passes through the centre of gravity of the displaced fluid, that point is called *the centre of buoyancy.*

663. Let v represent the volume of fluid displaced, and v' that of the body; D the density of the fluid, and D′ that of the body: the weights of the volume of displaced fluid, and

of the body will be respectively Dgv and $D'gv'$. If the body be supposed to rest in equilibrio, we shall have

$$Dgv=D'gv';$$

and if we suppose it to be entirely immersed, the volumes v and v' will be equal, and the densities D and D' must likewise be equal, in order that the equilibrium may be preserved.

But if the weight of the body be less than that of the fluid displaced, we shall have

$$Dgv>D'gv';$$

and the body will be urged upwards by a force equal to the difference $Dgv-D'gv'$.

If, on the contrary, we should have

$$Dgv<D'gv',$$

the body would tend downwards with a force equal to $D'gv'-Dgv$.

Of the Equilibrium, Stability, and Oscillations of Floating Bodies.

664. The propositions demonstrated in Arts. 662 and 663 establish two principles which serve as the basis of the theory of floating bodies; these principles are,

1°. *When a body is partially or totally immersed in a fluid, an equilibrium cannot subsist unless the centre of gravity and centre of buoyancy be situated upon the same vertical line.*

2°. *If an equilibrium be maintained, the weight of the body will be equal to that of the fluid displaced.*

The latter principle is frequently employed for the purpose of estimating the weight of a ship either with or without her cargo. For this purpose, we measure the capacity of the part immersed, and allow a weight of *one ton* for every 35 *cubic feet* which it contains. By taking the difference of the weights of the vessel with and without the cargo, the weight of the latter may be obtained. We can also arrive at the same result, by simply measuring the additional portion of the vessel immersed, when the cargo is introduced.

665. The horizontal surface of the fluid is called the *plane of floatation.*

666. If v denote the volume of fluid displaced, D its density, and g the intensity of the force of gravity, the weight P of the body ABC (*Fig.* 223), which floats upon the surface of the fluid, and is partially immersed, will be equal to Dgv.

667. When the floating body and fluid are both homogeneous, the centre of gravity of the part immersed will coincide with the centre of buoyancy.

668. The fluid and body being homogeneous, the centre of gravity G (*Fig.* 223) will be situated above the point O, the centre of buoyancy. For let g be the centre of gravity of that portion of the body which lies without the fluid: then, the centre of gravity G of the entire body will necessarily be situated upon the line gO, and between the points g and O; hence, it will be found above the point O.

669. But if the floating body be heterogeneous, it may happen that the centre of gravity of the entire body will lie below the centre of buoyancy. For by supposing the density of the lower part of the body to be very much greater than that of the upper portion, the centre of gravity of the entire body may be situated extremely near the lower surface: but the position of the centre of buoyancy depends only on the figure of the part immersed, since the density of the fluid is supposed uniform, and it may therefore be situated at a greater distance from the lower surface of the body than the centre of gravity of the entire mass.

Hence we conclude, that the centre of gravity of the floating body is sometimes situated above, and sometimes below, the centre of buoyancy.

670. When the body is but partially immersed, the weight of the immersed portion is less than that of the fluid displaced, and the equilibrium is maintained by the weight of that portion of the body which lies without the fluid: this weight is equal to the difference of the weights of the fluid displaced and of the part of the body immersed. If the weight of the body be increased, it will sink to a greater depth, until the weight of the additional quantity of fluid displaced shall be equal to the weight added.

671. Let us now suppose that a body floating upon the surface of a fluid (*Fig.* 224) is deranged in a very slight degree from its position of equilibrium, by the application of any force, and let us examine whether the body will tend to return to its original position, or, on the contrary, to deviate farther from it. Let ADB represent the immersed part of the body before derangement, and *ab*D that immersed after derangement: we suppose the new position of the body to be such, that the weight of the fluid displaced shall still be equal to the weight of the body, or that $ABD = abD$. The centre of gravity G may be regarded as fixed during the rotation, since the forces will tend to turn the system about that point, as though it were immoveable. The centre of buoyancy will not retain its position O, but will be found nearer to the portion CB*b*, which, by the rotation, has become immersed in the fluid: and if we suppose, for the sake of symplifying the question, that the body is divided symmetrically by the plane ABD, the centre of buoyancy will obviously be found in this plane after the derangement. Let *o* represent the centre of buoyancy in the deranged position, and through *o* and G let perpendiculars *oi* and G*k* be demitted upon the line *ab*. If an equilibrium subsist, the weight of the body and the upward pressure of the fluid will be equal and directly opposed.

The first condition will necessarily be satisfied, since we have supposed the volume of fluid displaced to remain unchanged: the second condition will be fulfilled when the points *i* and *k* coincide with each other: but if this coincidence should not take place, the point *i* may fall either to the right or to the left of the point *k*. In the first case, the pressure of the fluid applied at *o* and acting upwards, will evidently tend to restore the body to its primitive position, or to render the line DG vertical. But if the point *i* should fall to the left of *k*, this pressure would tend to turn the body in a contrary direction about the point G, and would thus cause it to deviate farther from its original position.

If the body, when deranged in a very slight degree from its position of equilibrium, should tend to resume its former position, the equilibrium is said to be *stable;* but if, on the contrary, it should tend to depart still farther from this position, the equilibrium is called *unstable;* when the body neither tends to return to its original position, nor to deviate farther from it, the equilibrium is said to be one of *indifference.*

672. By examining the directions of the pressures before and after derangement, we shall find that the lines OG and oi perpendicular to AB and ab respectively, are inclined to each other, and being contained in the same plane, they will intersect in some point m (*Fig.* 226).

This point is called the *metacentre;* and it appears from Art. 671, that when the point G is situated below m, the extremity k of the perpendicular Gk will fall to the left of the point i, and the equilibrium will be stable; but if the point G be situated above the point m, the extremity k of the perpendicular Gk will fall to the right of the point i, and the equilibrium will become unstable. If the points i and k coincide, the equilibrium becomes one of indifference.

673. Let it now be required to determine the position of the metacentre. This point being found upon the line connecting the centre of gravity and centre of buoyancy in the primitive position of the body, it will be sufficient to determine its distance from the point G, or the point O.

For this purpose we remark, that when the body is slightly inclined, the line AB (*Fig.* 226) which represents the profile of the plane of floatation in the primitive position, assumes a position inclined to the new plane of floatation ab in a certain angle ω, the portion ACa being at the same time withdrawn from the fluid, and the portion BCb being immersed. Hence, the immersed portions of the body in the two positions will be,

aCBD + ACa . . . . . in the primitive position,
aCBD + BCb . . . . . after the derangement.

But if γ, g, and g' represent the respective centres of gravity of the volumes aCBD, aCA, and bCB, the centre of gravity O of the volume ABRD will be found by dividing the line $g\gamma$ in the inverse ratio of the volumes aCBD and ACa; and in like manner, we may find the centre of gravity o of the volume abBD: thus, we shall obtain the proportions

$$\text{vol } a\text{CBD} : \text{vol } a\text{CA} :: \text{O}g : \text{O}\gamma \; . \; . \; . \; . \; . \; (400),$$
$$\text{vol } a\text{CBD} : \text{vol } b\text{CB} :: og' : o\gamma \; . \; . \; . \; . \; . \; (401);$$

but the second terms of these proportions are equal to each other: for, the floating body being supposed to displace the same quantity of fluid after it has been deranged as it did in its primitive position, the volumes ABRD and abRD will be equal to each other; and if from these equals we subtract the common part aCBD, there will remain the volumes aCA and BCb equal to each other. Hence, we deduce from the proportions (400) and (401),

$$\mathrm{O}g : og' :: \mathrm{O}\gamma : o\gamma;$$

which proves that the lines $g\gamma$ and $g'\gamma$ are cut proportionally by the right line Oo, which line is therefore parallel to gg'.

But, the derangement of the body being, by hypothesis, extremely slight, the line gg' may be considered as nearly coincident with the primitive plane of floatation; and since Oo is parallel to gg', this line may be regarded as parallel to the same plane.

674. To determine the value of Oo, we deduce, from the proportion (400),

$$\text{vol } a\text{CBD} + \text{vol } a\text{CA} : \text{vol } a\text{CA} :: \mathrm{O}g + \mathrm{O}\gamma : \mathrm{O}\gamma,$$

or,

$$\text{vol ABRD} : \text{vol } a\text{CA} :: g\gamma : \mathrm{O}\gamma.$$

But the similar triangles $gg'\gamma$ and O$o\gamma$ give

$$g\gamma : \mathrm{O}\gamma :: gg' : \mathrm{O}o;$$

and by comparing this proportion with the preceding, we obtain

$$\text{vol ABRD} : \text{vol } a\text{CA} :: gg' : \mathrm{O}o \;\ldots\ldots (402);$$

whence,

$$\mathrm{O}o = \frac{\text{vol } a\text{CA} \times gg'}{\text{vol ABRD}} \;\ldots\ldots (403).$$

675. Having determined the value of Oo, we can readily obtain that of Om (*Fig.* 227); for, the lines Om and om being respectively perpendicular to CA and Ca, the angles at C and m will be equal; and since these angles are exceedingly small, we may regard the triangles ACa and Omo as similar and isosceles: hence, we shall obtain the proportion

$$\mathrm{A}a : \mathrm{O}o :: \mathrm{C}a : m\mathrm{O};$$

and therefore,

$$m\mathrm{O} = \frac{\mathrm{O}o \times \mathrm{C}a}{\mathrm{A}a} \;\ldots\ldots (404).$$

676. To obtain the analytical expressions for O*o* and *m*O, we remark, that the plane of floatation AB (*Fig.* 228), which limits the immersed part of the body in its primitive position, is replaced by the plane *ab* after the derangement: these two planes, being intersected by a vertical plane perpendicular to their common intersection, will exhibit the section AC*a* represented in *Fig.* 226; and if we continue to draw other parallel vertical planes, we shall divide the solid included between the planes KAL, K*a*L (*Fig.* 228) into an infinite number of elementary laminæ parallel to the plane AC*a*.

But it is evident, that when the plane KAL, which in the primitive position of the body coincided with the surface of the fluid, shall have been detached from the surface, revolving around the line KL, each right line in this plane, as CA, will have described the sector of a circle; so that the sections of the solid included between the planes ALB and *a*L*b* (*Fig.* 228) by the system of parallel vertical planes, will be represented by the sectors AC*a*, A′C′*a*′, A″C″*a*″, &c. (*Fig.* 229). But if we assume the line of intersection KL as the axis of x, and place the axis of y in the plane KAL, the ordinates y will be the perpendiculars AC, A′C′, A″C″, &c. The infinitely small angle formed by the planes KAL and K*a*L being everywhere the same, let the arc described by a point at the distance unity from the line KL be expressed by ω: the arc described by the point A will then be determined by the proportion

$$1 : \omega :: \text{AC or } y : \text{arc A}a\,;$$

whence,

$$\text{arc A}a = \omega y \; \ldots\ldots \; (405).$$

This arc being multiplied by the half of the radius y, we shall obtain $\frac{1}{2}\omega y^2$ for the area of the sector AC*a*; and this area being multiplied by $\text{CC}' = dx$, the portion of the line KL intercepted between two consecutive sectors, we shall have for the volume of the solid

$$\text{A}aa'\text{A}' = \tfrac{1}{2}\omega y^2 dx,$$

which will express the element of the solid included between the planes KAL and K*a*L. Hence, we shall have (*Fig.* 226)

$$\text{vol AC}a = \tfrac{1}{2}\omega \textstyle\int y^2 dx \; \ldots\ldots \; (406).$$

Such will be the analytical expression for the second term of the proportion (402).

To determine the value of the third term, we remark that the line Cg (*Fig.* 226) being the distance of the axis KL (*Fig.* 229) from the centre of gravity of the solid KaLA, we shall determine this distance, by dividing the sum of the moments of the elementary solids by the volume KaLA.

If we consider the elementary sector ACa (*Fig.* 229), the centre of gravity g of this sector will be found upon the radius CR=CA (*Fig.* 230), at a distance from the point C (Art. 184) expressed by

$$\tfrac{2}{3}\mathrm{CR}\times\frac{\text{chord Aa}}{\text{arc Aa}};$$

but the angle C being supposed extremely small, the arc Aa may be regarded as equal to the chord; and since CR is equal to CA or y (*Fig.* 228), the preceding expression will give $\tfrac{2}{3}y$ for the distance of the centre of gravity from the axis KL. Multiplying the elementary solid $\tfrac{1}{2}\varpi y^2dx$ by this distance, the moment of this solid with reference to this axis KL will become $\tfrac{1}{3}\varpi y^3dx$: thus, we shall have

$\int\tfrac{1}{2}\varpi y^2dx$=the sum of the elementary solids,

$\int\tfrac{1}{3}\varpi y^3dx$=the sum of the moments of the elementary solids:

and from the property of the moments, the distance Cg of the centre of gravity of the small solid CAa (*Fig.* 226), or KALa (*Fig.* 229), will be expressed by

$$\mathrm{C}g=\frac{\int\frac{1}{3}\varpi y^3dx}{\int\frac{1}{2}\varpi y^2dx}:$$

the quantity ϖ being constant, this expression may be reduced to

$$\mathrm{C}g=\frac{2\int y^3dx}{3\int y^2dx}.$$

677. The value of Cg will result from the integrations here indicated; and that of Cg′ (*Fig.* 226) may be obtained in a similar manner; but, if the floating body be symmetrical with respect to a vertical plane passing through the axis KL, as will always happen in the case of a ship, we shall have

$$\mathrm{C}g=\mathrm{C}g',$$

and therefore,

$$gg'=2\mathrm{C}g=\frac{4\int y^3dx}{3\int y^2dx}\;.\;.\;.\;.\;.\;(407).$$

The volume of the part immersed, which likewise enters into the equation (403), can be calculated directly, when the figure of the vessel is supposed known. Let this volume be denoted by V, and let its value and those of the volume ACa and gg', given in equations (406) and (407), be substituted in equation (403): we shall thus obtain

$$Oo=\frac{2\omega\int y^3dx}{3V}:$$

and lastly, by substituting in equation (404) this value, and that of the arc Aa, given by equation (405), replacing Ca by y, we find

$$mO=\frac{2.\int y^3dx}{3V}.$$

Such is the formula expressive of the distance of the metacentre from the centre of buoyancy.

678. When the floating body is homogeneous, and of such figure that its parallel sections will be similar, we may readily determine the position of the metacentre, without the necessity of performing an integration. For let a^2 represent the area of the section AEB (*Fig.* 231), which is supposed to have been determined by direct measurement, and let b represent the half-breadth CA of this section: the half-breadths of the sections A′E′B′, A″E″B″, &c. will be represented by C′A′, C″A″, &c. or by the ordinates y of the curve KAL. These sections being by hypothesis similar figures, they will be proportional to the squares of their homologous sides; and hence, we shall have

$$\text{section AEB} : \text{section A}'\text{E}'\text{B}' :: \text{AC}^2 : \text{A}'\text{C}'^2,$$

or,

$$a^2 : \text{section A}'\text{E}'\text{B}' :: b^2 : y^2;$$

whence,

$$\text{section A}'\text{E}'\text{B}'=\frac{a^2y^2}{b^2}.$$

The distance CC′ between two consecutive sections being denoted by dx, we shall have

$$\frac{a^2y^2dx}{b^2}$$

for the expression of the elementary solid.

679. Let g represent the centre of gravity of the section AEB, which, in consequence of the symmetry of the figure, will be found on the vertical CE. The centre of gravity of this section having been determined, let its distance from the surface of the fluid be denoted by n: we shall then have, from the similarity of figures,

$$b:y::n:\left\{\begin{matrix}\text{the distance of the centre of gravity of the}\\ \text{section A}'\text{E}'\text{B}'\text{ from the surface of the fluid}\end{matrix}\right\}=n';$$

whence,

$$n'=\frac{ny}{b}.$$

Multiplying this distance by the elementary solid, we shall obtain for the moment of this solid, taken with reference to the surface of the fluid,

$$\frac{ny}{b}\times\frac{a^2y^2}{b^2}dx:$$

and therefore the expression $\frac{na^2}{b^3}\int y^3dx$ will represent the sum of the moments of the elementary solids taken with reference to the surface of the fluid. This sum being equal to the product of the volume V of the solid immersed by the depth HG of its centre of gravity, if this depth be denoted by G, we shall have

$$\text{VG}=\frac{na^2}{b^3}\int y^3dx;$$

whence,

$$\text{G}=\frac{na^2}{\text{V}.b^3}\int y^3dx.$$

But it has been shown (Art. 677) that the distance mO of the metacentre from the centre of buoyancy is given by the formula

$$m\text{O}=\frac{2\int y^3dx}{3\text{V}};$$

and if we compare these two expressions, we shall find

$$\text{G}:m\text{O}::\frac{na^2}{\text{V}.b^3}\times\int y^3dx:\frac{2}{3}\times\frac{\int y^3dx}{\text{V}},$$

or,

$$\text{G}:m\text{O}::3na^2:2b^3;$$

whence,

$$mO=\frac{2b^3G}{3na^2}\ \ldots\ldots\ (408).$$

680. For the purpose of applying this formula, let it be required to find the metacentre of a rectangular parallelopiped ML. Let AF represent the intersection of the body by the surface of the fluid (*Fig.* 232), supposed parallel to the base NL. The depth AN, to which the body must be immersed in order that it may be sustained in equilibrio, will depend on the weight of the parallelopiped and the density of the fluid (Art. 664): this depth may be considered as determined by experiment: the quantity a^2, which represents the section BN, and which will be constant for all parallel sections, will be determined immediately; for we have

$$a^2=AB\times CE.$$

Again, the semi-breadth of the section being equal to $\frac{1}{2}AB$, there results

$$b=\tfrac{1}{2}AB=AC;$$

and since the centres of gravity of all the sections are equally distant from the surface of the fluid, the centre of gravity of the fluid displaced will be situated at the same distance; so that we shall have

$$n=G=\tfrac{1}{2}CE.$$

By substituting these values in formula (408), the distance of the metacentre m from the centre of buoyancy will be found equal to

$$mO=\frac{2AC^3}{3AB\times CE}$$

or, by reduction,

$$mO=\frac{AC^2}{3CE}.$$

For example, if the semi-breadth of the parallelopiped be supposed equal to 9 feet, and the depth of the part immersed 4 feet, we shall find the height of the metacentre above the centre of buoyancy equal to 6¾ feet; if, therefore, we subtract from this height, 2 feet, the depth of the centre of buoyancy, there will remain 4¾ feet, for the height of the metacentre above the surface of the fluid. Hence, the centre of gravity of the parallelopiped should not be more than $4\frac{1}{3}$ feet above the surface of the fluid, if we wish the equilibrium to be of the stable kind.

681. As a second example, let us consider a vessel whose vertical sections below the surface of the fluid are equal right-angled isosceles triangles, such as AEB (*Fig.* 233).

If the perpendicular EC be demitted upon the base, the triangle AEC will likewise be isosceles, and the height EC will therefore be equal to one-half the base AB: thus, the quantities which enter into the formula (408) will be, in the present case,

$a^2 = \text{area of the triangle AEB} = \text{AC}^2,$
$n = \text{G} = \frac{1}{3}\text{CE},$
$b = \text{AC} = \text{CE};$

consequently, by substituting these values in formula (408), it will reduce to

$$m\text{O} = \tfrac{2}{3}\text{CE}:$$

and if from this value we subtract that ot the distance of the point O below the surface of the fluid which is equal to $\frac{1}{3}$CE, there will remain $\frac{1}{3}$CE for the distance of the metacentre above the surface of the fluid. Hence, in a prismatic vessel whose vertical sections are right-angled isosceles triangles, the metacentre will be found at a distance above the surface of the fluid equal to the distance of the centre of buoyancy below the surface.

682. If we suppose the body to be slightly deranged from a position of stable equilibrium, and conceive the resultant of all the upward pressures of the fluid to be applied on its line of direction, at the metacentre, we can determine the circumstances of oscillation of this body about the centre of gravity, by a method entirely analogous to that employed in considering the motion of the compound pendulum. For this purpose, let the origin of co-ordinates be placed at the centre of gravity, and let the proper value of $y_{,}$ be substituted in formula (337), which may be put under the form (338)

$$\frac{d\omega}{dt} = \frac{gy_{,}}{k^2 + a^2}.$$

This formula admits of simplification in the present case, from the consideration that the oscillations are performed about the centre of gravity; and the general expression of the moment of inertia $M(k^2+a^2)$ is therefore reduced to Mk^2; hence, we obtain

$$\frac{d\omega}{dt}=\frac{gy_{,}}{k^2}\,\ldots\ldots\,(409).$$

This equation, when integrated, will serve to determine the angular velocity, and the time of performing a complete oscillation.

683. To determine the value of $y_{,}$, which represents the perpendicular distance from the axis passing through the centre of gravity, about which the oscillations are performed, to the line of direction of the upward pressure, we remark, that the distance of the metacentre from the centre of buoyancy O is expressed by

$$\frac{2\int y^3dx}{3V}.$$

Let this distance be denoted by A, and the distance GO (*Fig.* 234) by B; we shall then have

$$Gm=A+B;$$

or, since the point G may fall above O, we may likewise have

$$Gm=A-B;$$

hence we may comprise the two cases under the double sign, by writing

$$Gm=A\pm B.$$

If the angle LmG (*Fig.* 234), formed by the vertical mL with the new direction of the line GO, be represented by θ, we shall have the relation

$$GL=Gm\sin\theta;$$

or, replacing the sine by the arc, since the arc is supposed extremely small, and substituting the value of Gm, this equation will become

$$GL=(A\pm B)\theta;$$

and by introducing this value of $y_{,}$ in formula (409), we shall obtain

$$\frac{d\omega}{dt}=\frac{g(A\pm B)\theta}{k^2}.$$

684. But the angular velocity ω being that which corresponds to the arc θ described with a radius unity, this velocity will be expressed by $\frac{d\theta}{dt}$; and since the arc θ (*Fig.* 234) is a decreasing function of the time t, $d\theta$ should be affected with the negative sign; hence,

$$\omega=\frac{-d\theta}{dt}\;\ldots\ldots\;(410).$$

Multiplying the corresponding terms of these equations together dt will disappear: and there will result

$$\frac{g(\mathrm{A}\pm\mathrm{B})}{k^2}\theta d\theta+\omega d\omega=0.$$

Putting, for brevity,

$$\frac{g(\mathrm{A}\pm\mathrm{B})}{k^2}=\mathrm{E}\;\ldots\ldots\;(411),$$

and multiplying by 2, we obtain

$$2\mathrm{E}\theta d\theta+2\omega d\omega=0.$$

Integrating, we have

$$\mathrm{E}\theta^2+\omega^2=\mathrm{C}:$$

whence,

$$\omega=\sqrt{(\mathrm{C}-\mathrm{E}\theta^2)}.$$

Substituting this value in equation (410), we obtain

$$dt=-\frac{d\theta}{\sqrt{(\mathrm{C}-\mathrm{E}\theta^2)}};$$

or, by reduction,

$$dt=-\frac{d\theta}{\sqrt{\mathrm{E}}\sqrt{\left(\frac{\mathrm{C}}{\mathrm{E}}-\theta^2\right)}};$$

and, by integration,

$$t=\frac{1}{\sqrt{\mathrm{E}}}\,\mathrm{arc}\left(\cos=\frac{\theta\sqrt{\mathrm{E}}}{\sqrt{\mathrm{C}}}\right)+\mathrm{C}':$$

from which we deduce

$$\frac{\theta\sqrt{\mathrm{E}}}{\sqrt{\mathrm{C}}}=\cos\left[(t-\mathrm{C}')\sqrt{\mathrm{E}}\right]:$$

and, finally,

$$\theta=\frac{\sqrt{\mathrm{C}}\,.\cos\left[(t-\mathrm{C}')\sqrt{\mathrm{E}}\right]}{\sqrt{\mathrm{E}}}.$$

685. When E is negative, the value of θ becomes imagin-

ary, and the oscillatory motion cannot take place; but in order that E may be negative, the first member of equation (411) must likewise be negative; and consequently,

$$A \pm B = \text{a negative quantity}:$$

this case occurs when B exceeds A, and is affected with the negative sign; and since $A \pm B$ represents the distance of the centre of gravity from the metacentre, it follows that the metacentre will then be situated below the centre of gravity, and the equilibrium will be unstable. On the contrary, if $A \pm B$ be positive, the metacentre will be found above the centre of gravity, the value of E will be positive, and the values of θ and ω will be real: thus, the oscillations can be performed, and the equilibrium will be of the stable kind.

686. The time of oscillation being determined by a method entirely similar to that employed in investigating the circumstances of motion of the compound pendulum, we may conclude that this time will be independent of the extent of the arc through which the oscillations are performed, provided the arcs be extremely small.

Specific Gravity—Hydrostatic Balance—Hydrometer.

687. Let P represent the weight of a body M: if this body be immersed in a fluid, the buoyant effort exerted by the fluid will tend to support the body, and the force P′ necessary to sustain it will be less than P, that required previous to the immersion, by a quantity equal to the weight of the fluid displaced.

For example, if M be supposed a sphere of lead whose weight is equal to eleven pounds, and if it be found to weigh but ten pounds when immersed in water, we should conclude that the weight of an equal volume of water would be one pound; and therefore that the weight of lead was to that of water as eleven to one.

688. The *specific gravity* of any substance is the ratio between its weight and the weight of an equal volume of some other substance assumed as the standard.

Thus, in the preceding example, if water be adopted as the standard of comparison, the weight of the sphere of lead being eleven times greater than that of an equal volume of water, the specific gravity of lead will be represented by the number 11.

The density of a body has been defined (Art. 161) to be the ratio between the quantity of matter contained in the body and that contained in an equal volume of some other substance assumed as the standard; and since the weights of bodies are proportional to the quantities of matter which they contain, it follows that the ratio of the weights of two bodies will be equal to the ratio of their quantities of matter. Hence, the number expressing the specific gravity of a body will be the same as that which expresses its density, provided we refer the density and specific gravity to the same substance as a standard.

In practice, it is usual to adopt water as the standard in determining the specific gravities of solids and incompressible fluids; and for the purpose of rendering the comparison more exact, the water is first deprived, by distillation, of any impurities which it may contain. The specific gravities of gases and vapours are generally referred to that of atmospheric air.

689. The dimensions of all bodies being more or less affected by changes of temperature, it becomes necessary to adopt a standard temperature, at which experiments for the determination of specific gravities may be performed. A convenient temperature for this purpose is that corresponding to 60° of Fahrenheit's thermometer, it being easily obtained at all times: and the tables of specific gravities are usually calculated for this temperature. When circumstances will not permit the experiments to be performed at the standard temperature, the results obtained must be reduced to this temperature, by introducing a correction for the change of volume which the substance would undergo if reduced to the standard temperature. This correction is readily applied when the law of dilatation has been previously ascertained.

690. If we wish to determine the specific gravity of a fluid, as olive-oil, we may immerse successively the same solid in water and in this fluid; we shall thus be enabled to determine the weights of equal volumes of the two fluids; and a

comparison of these weights will give the specific gravity of the oil. For example, if the sphere of lead weighing eleven pounds have its weight reduced to 10.085 lb. when immersed in oil, the weight of the fluid displaced would be equal to 0.915 lb.; and since the weight of an equal bulk of water was found equal to 1 lb., we shall obtain $\frac{0.915}{1}=0.915$, for the ratio of the weights of equal bulks of the two fluids: this number will therefore represent the specific gravity of oil.

From the preceding remarks, we may infer that if two bodies of unequal volumes, suspended from the arms of a balance, sustain each other in vacuo, the equilibrium will not be maintained when the bodies are similarly suspended in the atmosphere; the weight of the larger body being most supported by the buoyant effort of the atmosphere.

691. The instrument usually employed for determining with accuracy the specific gravities of bodies, is the *hydrostatic balance.* This consists merely of a delicate balance, having a small hook attached to one of its scales, by means of which the body can be suspended, for the purpose of determining its weight when immersed in a fluid. The body is connected with the hook by a hair or slender thread, whose weight is inconsiderable.

When we wish to determine the specific gravity of a solid, we place it in the scale to which the hook is attached, and add weights in the opposite scale until an equilibrium is produced. The weights thus added will represent the weight of the body in air. The body is then attached to the hook and immersed in water; and the weight necessary to be placed in the opposite scale to produce an equilibrium will give its weight in water: the difference between the weights in air and water will be equal to the weight of an equal volume of water, and by comparing this difference with the weight in air, we shall obtain the specific gravity of the substance under consideration.

This process is slightly inaccurate; since the buoyant efforts exerted by the atmosphere upon the body when immersed in it, and upon the weights introduced into the opposite scale, have been neglected. But as the density of the atmosphere is very small, this omission will not affect the results materially.

When the given substance is soluble in water, we determine its specific gravity with reference to some fluid in which it is insoluble, and then compare the specific gravities of the two fluids. If the body be lighter than water, we can connect it with a heavier body, which will cause it to sink. Then, having the weights of the heavier and lighter bodies, and that of the compound in air, and having ascertained the loss of weight sustained by the heavier body and the compound when immersed, we can readily deduce the weight of the fluid displaced by the lighter.

The specific gravity of a fluid may be determined by weighing successively the same body in this fluid and in water, and comparing the weights of the equal volumes displaced. Or it may be ascertained by weighing the same vessel when filled with water, and with the fluid under consideration; these weights, being diminished by that of the vessel when empty, will give the relation between the specific gravity of the fluid and that of water.

692. *The hydrometer* is an instrument usually designed to determine approximatively the specific gravities of fluids. It is composed of a cylinder of glass or metal, to the lower extremity of which a cup is attached loaded with shot or mercury, and terminated at top by a slender graduated wire.

When the hydrometer is plunged into a fluid, the weight with which its lower extremity is loaded causes it to assume a vertical position, and it sinks to a greater or less depth, according to the specific gravity of the fluid. Hence, that division on the graduated stem which corresponds to the surface of the fluid will serve to indicate the specific gravity of the fluid.

For example, if the hydrometer be immersed in distilled water whose temperature corresponds to 60° Fahrenheit, the surface of the water will intersect the stem at a certain division, which we shall suppose to be that marked 10: if plunged in wine, it will sink deeper, say to the 11th, 12th, or 13th division; and if in brandy, to a still greater depth, the division indicated being dependent on the quantity of alcohol which the brandy contains.

The use of this instrument evidently depends upon the principle, that when a body is immersed in a fluid, a portion of its weight equal to that of the fluid displaced will be supported by the buoyant effort of the fluid: thus, the heavier the fluid, the less the depth to which the hydrometer will sink.

693. The hydrometer, as improved by Nicholson, will serve to determine the specific gravities of solids or liquids. The instrument consists of a hollow copper ball A (*Fig.* 235), to the lower part of which is attached a brass cup of sufficient weight to maintain the hydrometer in a vertical position when immersed in a fluid. The upper part of the ball carries a slender wire D, which supports a small dish C destined to receive the weights. The weight of the hydrometer is such that the addition of 500 grains in the dish C will just sink the instrument in distilled water, at the temperature 60°, until the surface of the water intersects the stem at its middle point D. If, therefore, a body be placed in the dish C, and weights be added until the point D shall correspond to the surface of the water, the difference between 500 grains and the weights added will express the weight of the body. The body being then transferred to the lower dish B, it will be found necessary to place additional weights in the dish C, in order to sink the hydrometer to the same depth: these additional weights will be equal to the loss of weight sustained by the body when immersed. Hence, the specific gravity of the solid may be readily determined.

When we wish to determine the specific gravity of a fluid with this hydrometer, we immerse the instrument successively in distilled water and in the given fluid, and ascertain the weights necessary to be added in each case to the dish C, in order to sink it to the same level. Then, the known weight of the instrument added to the weights introduced into the upper dish will give the weight of the fluid displaced. Thus, we can compare the weights of equal volumes of the two fluids.

Of the Pressure and Elasticity of Atmospheric Air.

694. The weight of the atmosphere was first recognised by Galileo. Torricelli, his pupil, demonstrated the existence of this weight by the following experiment. Let AB (*Fig.* 236) represent a glass tube, 3 feet in length, filled with mercury, closed at the lower extremity and open at the upper: let the finger be applied to the open extremity, and let the tube be inverted, and its open extremity plunged in the basin of mercury: on withdrawing the finger, the mercury will be found to descend in the tube, leaving a certain portion of it BE (*Fig.* 237) unoccupied. If the experiment be tried with tubes of different lengths or different diameters, the height of the column of mercury sustained in the tube will be found, in each case, to be about 29 or 30 inches above the level of the fluid in the basin. This column of mercury is sustained by the pressure of the atmosphere, arising from its weight; which pressure, being exerted upon the surface CD, is sufficient to counterbalance the weight of the column.

If the experiment be performed with fluids of different densities, the heights at which they will be supported will be found to differ: thus, if the fluid be water, whose density is to that of mercury as 1 to $13\frac{1}{2}$, the height of the column will be found equal to $30 \text{ in.} \times 13\frac{1}{2} = 34$ feet, nearly; the weight of such column being equal to the weight of the column of mercury.

695. The operation of the common siphon is also to be referred to the pressure of the atmosphere.

The siphon is a bent tube having its two branches of unequal lengths. The shorter branch EF (*Fig.* 238) being plunged into the fluid contained in the vessel ABCD, and the air being withdrawn from the siphon, the pressure of the atmosphere exerted upon the surface BC will cause the fluid to rise in the siphon; and if the height of the point F be less than that at which the atmospheric pressure can sustain the given fluid, it will pass into the longer branch, and will be delivered at the point C. The current having commenced in the siphon, it is maintained in consequence of the superior weight of the fluid in the longer arm overcoming, in part, the pressure of the atmosphere at the point C, and thus permitting the equal pressure of the atmosphere exerted upon the surface BC to force the fluid up the shorter branch. Hence, it is obvious that the point C must always be below the surface of the fluid in the reservoir ABCD, in order that the siphon may be effective.

696. Air is an elastic fluid, which is susceptible of being compressed into spaces which bear to each other the inverse ratio of the forces applied.

This may be established experimentally as follows: Let A′BCE (*Fig.* 239) represent a curved tube closed at E and open at A′: let mercury be introduced into the tube until it shall stand at the same level CC′ in the two branches: the air contained in the space CE will then be of the same density as the exterior air. If mercury be now poured into the tube until the part ABCD be entirely filled, the length AB being equal to 30 inches, the column of air DE will be found reduced to one-half its original bulk CE: if mercury be again introduced until it extend from A′ to *d*, the length A′*b* being equal to 60 inches, the volume of air will be found reduced to a space $Ed = \frac{1}{3}CE$.

This experiment establishes the law of compressibility; for, before the introduction of the mercury, the air contained in the space CE, being pressed by the weight of the atmosphere, must support a pressure equivalent to 30 inches of mercury. When the same volume of air is caused to sustain the additional pressure of a column of mercury AB=30 inches, it is reduced to one-half its original bulk; and by the further addition of 30 inches, the air is reduced to one-third of this bulk. Thus, it appears, that the spaces occupied by the same mass of air are inversely proportional to the pressures applied; and since the densities of the air are inversely proportional to the spaces occupied by the same mass, it follows that the densities will be in the direct ratio of the pressures.

If the mercury be withdrawn from the tube, the air will expand and occupy the same space as it did previous to compression.

Of Pumps for raising Water.

697. The *pump* is a machine employed for the purpose of raising water. There are three principal kinds of pumps, viz. the *sucking* pump, the *lifting* pump, and the *forcing* pump.

The sucking pump, represented in *Fig.* 240, consists of two tubes ABDC and DCHL, of unequal diameters, connected together; the first of these is called the sucking pipe, and the second the body of the pump. Within the body of the pump, an air-tight piston MN, having a valve opening upwards, is moved through the space MH, which is called the play of the piston. At the lower extremity of the body of the pump, a second valve *k*, called the sleeping valve, is placed, which likewise opens upwards.

The lower extremity AB of the sucking pipe being immersed in a reservoir containing water, and the piston MN being raised from the position MN to HL, the air contained in the space CN will expand and fill the space CL, its density and elastic force being both diminished: at the same time, the air contained in the pipe AD, having a density equal to that of the exterior air, will, in virtue of its elasticity, exert upon the valve *k*, a stronger pressure than that arising from the elasticity of the rarefied air contained in the space CL: hence, the valve *k* will be forced open, and the air contained in the interior of the pump will acquire a density that is uniform throughout, but less than that of the exterior air: then the pressure exerted upon the surface of the water AB being less than that exerted by the atmosphere upon the surface at other points of the reservoir, the water will rise in the sucking pipe to the level A′B′, such that the weight of the column A′B, together with the pressure of the rarefied air contained in the pump, shall be equal to the pressure of the exterior air. The densities of the air in the body of the pump and in the sucking pipe having become equal, the valve *k* closes by its own weight.

The piston being then depressed from the position HL to MN, the air contained in the space CL will be compressed into the space CN, and its density and elastic force will become greater than those of the air contained in the sucking pipe: the pressure on the upper surface of the valve *k* being now greatest, this valve will continue closed during the descent of the piston, and will intercept the communication between the sucking pipe and body of the pump: hence, the density of the air in the sucking pipe will remain unchanged, and the water will retain the level A′B′. When the piston shall have regained the position MN, it will have compressed into the space CN, not only the quantity of air originally contained in CN, but likewise that portion which was introduced into the body of the pump from the sucking pipe. The density of the air contained in the space CN will therefore exceed that of the exterior air, and its elastic force will open the valve I: the air contained in CN will thus be restored to its original density. The piston being raised a second time, the air in MD will be again rarefied, a portion of that contained in A′D will pass into the body of the pump, and the equilibrium will be restored by the water rising to a new level A″B″.

The same operation being repeated, the water will rise through the valve *k* into the body of the pump, will pass through the valve I in the piston, and will finally be delivered by the spout QR.

698. We will next examine the mechanism of the lifting pump. In this pump, the piston MN (*Fig.* 241) is situated below the fixed valve *k*, and being depressed from the position MN to HL, is supposed to pass below the surface *a′b′* of the water contained in the reservoir: the piston contains a valve opening upwards, through which the water passes, regaining its level *a′b′*. The piston being then elevated, the column of water *a′*L, which rests upon its superior base, being prevented from returning through the valve, will be raised through a height equal to the play of the piston, and will occupy the space *a*N: at the same time, a vacuum being formed below the piston, the water will be compelled to follow the piston in its motion by the pressure of the atmosphere on the surface of the water in the reservoir. But the air contained in the space *a′*D being compressed by the elevation of the

piston, its elastic force will become greater than that of the exterior air, and the valve k will open, restoring the air below k to its original density. The circumstances will then be the same as before the first stroke of the piston, with the exception that a portion of water has passed above the piston. When the piston is again depressed, the column of water aN, which rests upon it, will also descend, and the air contained in the space Cb will therefore be rarefied. The descent of the water will continue until the elastic force of the rarefied air contained between the valve k and the surface of the water, together with the weight of the column of water raised, shall be equal to the pressure of the atmosphere: the valve in the piston will then open, and an additional quantity of water will pass above the piston. By repeating the process, a certain portion of water will pass above the piston at each stroke; and reaching the valve k, will pass into the body of the pump, and may be delivered at any height.

699. The forcing pump is a combination of the sucking and lifting pumps. In this pump, the piston MN (*Fig.* 242) is without a valve, but the lateral pipe HE is provided with one at I, opening upwards; and there is a sleeping valve at L, as in the sucking pump. The piston being raised, the water rises into the space MCDEF, for the reasons assigned in describing the sucking pump: when the piston is depressed, the water is forced through the valve I into the tube HG; and by continuing the process, it may be delivered at any height.

700. If the dimensions of the sucking pump be improperly chosen, it may happen that the water will rise only to a certain height. For the purpose of discovering in what cases this will occur, we shall simplify the question, by supposing the pump to be of uniform bore throughout. Let the water be supposed to have been raised to the level ZX (*Fig.* 243), and the piston to move through the space ML: call

a=LN, the play of the piston,

b=LB, the height of the piston at its greatest elevation above the surface of the water contained in the reservoir,

x=the distance LX.

When the piston is raised from the position MN to HL, the air which was previously contained in the space ZN will occupy the space ZL, and its elasticity will therefore be diminished in the ratio of LX to NX; so that if R represent the elastic force of the air contained in the space NZ, and R′ the elastic force of the rarefied air contained in LZ, we shall have

$$\mathrm{LX} : \mathrm{NX} :: \mathrm{R} : \mathrm{R}';$$

or,

$$x : x-a :: \mathrm{R} : \mathrm{R}':$$

whence,

$$\mathrm{R}'=\mathrm{R}\frac{x-a}{x}.$$

But the air contained in the space NZ being of the same density with the exterior air, its elastic force will be properly measured by the weight of a column of water whose base c is equal to the surface MN, and whose height is equal to 34 feet. Let this height be denoted by h; the density of water being supposed equal to unity, and the force of gravity being denoted by g, we shall have

$$\mathrm{R}=chg.$$

This value, substituted in the preceding equation, gives

$$\mathrm{R}'=\frac{x-a}{x}chg.$$

But it is evident that when an equilibrium subsists, the elastic force of this rarefied air, together with the weight of the column of water BZ, must be just sufficient to counterbalance the pressure of the atmosphere, which tends to produce the ascent of the water. The weight of the column of water ABXZ will be expressed by $gc\times\mathrm{BX}$, or $gc\times(b-x)$; and the pressure exerted by the atmosphere will be expressed by the column gch; hence, we shall have, in case of an equilibrium,

$$\frac{x-a}{x}gch+(b-x)gc=gch;$$

or, by suppressing the common factor gc,

$$\frac{x-a}{x}h+b-x=h.$$

But, if it were required that the water should rise above the level ZX, it would then be necessary that the atmospheric pressure should exceed that arising from the weight of the column ZB, and the elastic force of the air contained in the space ZL: we shall consequently have

$$\frac{x-a}{x}h+b-x<h.$$

Let z represent the excess of the second member of this inequality; then

$$\frac{x-a}{x}h+b-x+z=h;$$

or, by reduction,

$$-ah+bx-x^2+zx=0:$$

whence,

$$x=\frac{b+z}{2}\pm\sqrt{\left[\left(\frac{b+z}{2}\right)^2-ah\right]}.$$

If we make $z=0$, the water will cease to rise, and we shall then have

$$x=\frac{b}{2}\pm\sqrt{\left(\frac{b^2}{4}-ah\right)}.$$

These two values of x will be real so long as $\frac{b^2}{4}$ exceeds ah: if, therefore, this condition be fulfilled, there will be two points at which the water will stop: but if, on the contrary, ah should exceed $\frac{b^2}{4}$, the values of x will become imaginary, and there can be no point at which the water will cease to rise. Such is the condition requisite to ensure the effective performance of the sucking pump.

701. With the lifting pump, the water can be raised to any height, provided sufficient force be applied to the piston. For, let the water be supposed to have risen to the level EF (*Fig.* 241), the water in the reservoir standing at the level *ab*, above the piston. Then, the column included between the surfaces *ab* and MN being supported by the pressure of the contiguous fluid, the piston MNP will be loaded only with the weight of the column extending from *ab* to EF.

702. But if the level of the fluid in the reservoir be supposed at *a′b′* below the piston, the weight P of the column of water included between MN and *a′b′* must be supported by

the pressure of the atmosphere exerted upon the surface of the water in the reservoir. Hence, the pressure of the atmosphere exerted upon the upper base of the piston, through the column EN, will exceed that which is exerted upon the lower base through a'N, by the weight of the column a'N; for this weight counteracts in part the pressure exerted by the atmosphere upon the water in the reservoir. Thus, the piston MN being urged downwards by the weight of the column MF situated above it, and likewise by the difference of the atmospheric pressures, which is equal to the weight of the column a'N, the effect will be the same as though the piston supported a column of water whose base is MN, and whose altitude is equal to the distance between the levels $a'b'$ and EF.

It thus appears, that with a sufficient effort, the water may be raised to any height by the lifting pump, the fixed valve k being supposed near the surface of the water.

The same principles will serve to estimate the force necessary to raise the water in the sucking pump.

Of the Air-pump.

703. In examining the properties of various substances, it is frequently necessary to withdraw them from the action of the atmosphere, and it therefore becomes desirable that we should have it in our power to exhaust the air from a vessel in which the substance has been deposited. This vessel is called the receiver, and is usually constructed of a transparent substance, such as glass, in order that we may have an opportunity of observing the effects produced on the substance under consideration by the withdrawal of the atmospheric air.

704. The machine employed to exhaust the air is called an *air-pump*, and the term *vacuum* is applied to the space from which the air has been extracted.

705. The general principles upon which the operation of this machine depends, will be readily understood by a reference to *Fig.* 244. A represents a section of the glass receiver which rests upon the plate BC, the lower edge of the receiver and the plate being ground exactly plane, so that their con-

tact may be as perfect as possible. The edge of the receiver being previously smeared with a little sweet oil, the air will be effectually prevented from penetrating between the receiver and plate.

The plate BC is perforated by a cavity DE, which communicates with the cylindrical barrel CF, in which an air-tight piston, having a valve opening upwards, is worked by means of a handle H. At the bottom of the barrel is placed a second valve E, likewise opening upwards.

706. Let it now be supposed that the piston has been depressed until it has reached the valve E; the air in the receiver, barrel, and communicating pipe being of the same density as the exterior air, and the valves being closed by their own weight. Then, if a force be applied to raise the piston, the valve P will remain closed, and a vacuum would be left between the piston and the valve E, provided the weight of the valve E were sufficient to overcome the pressure exerted upon its under surface by the elastic force of the air contained in the receiver and communicating pipe: this, however, not being the case, the valve E will be forced open, and a portion of the air contained in the pipe and receiver will pass into the barrel, until the density of the air becomes uniform throughout. This effect will continue until the piston has reached its highest position, and the valve E will then close by its own weight. The piston being then depressed, the valve E will remain closed, and the air contained in the barrel being compressed into a smaller space, its elastic force will be increased, will become greater than that of the exterior air, and will finally overcome the weight of the valve P, causing it to open, and thus reducing the density of the air contained in the barrel to an equality with that of the exterior air: this effect will only cease when the piston has been forced to the bottom of the barrel.

It thus appears that by a single ascent and descent of the piston, a portion of air has been withdrawn from the receiver and pipe of communication. The portion withdrawn will obviously bear the same ratio to the quantity originally contained in the receiver and pipe that the capacity of the barrel bears to the sum of the capacities of the barrel, pipe, and receiver. By a repetition of the same process, a second quantity can be withdrawn, and the operation may be continued until the exhaustion has been carried to the desired extent.

707. Since the quantity of air withdrawn at each ascent and descent of the piston forms but a part of that previously contained in the receiver and pipe, it is obvious that a perfect vacuum can never be produced by the operation of the pump. The weight of the lower valve likewise opposes an obstacle to the entire exhaustion; for, whenever the air contained in the receiver and pipe shall have had its elastic force so far reduced as to be incapable of raising the valve E, the pump will necessarily cease to exhaust. This difficulty may, however, be obviated, by causing the valve E to open by means of a mechanical connexion with the piston.

708. As it is frequently necessary to produce a very high degree of exhaustion, it becomes interesting to ascertain the density of the air remaining in the receiver after any given number of strokes of the piston; and since the portion withdrawn at each double stroke bears a constant relation to that remaining, this density may be readily estimated. Thus, if we denote by b, p, and r the respective capacities of the barrel, pipe, and receiver, and by d the original density of the air, we shall have the proportion

$b+p+r : p+r :: d : d\frac{p+r}{b+p+r}$ =density after the first double stroke.

In like manner,

$b+p+r : p+r :: d\frac{p+r}{b+p+r} : d\left(\frac{p+r}{b+p+r}\right)^2$ =density after the second double stroke.

And generally,

$b+p+r : p+r :: d\left(\frac{p+r}{b+p+r}\right)^{n-1} : d\left(\frac{p+r}{b+p+r}\right)^n$ = density after the nth double stroke.

For the purpose of illustrating the rate of exhaustion, we will suppose that the capacity of the barrel is one-fourth of the sum of the capacities of the receiver and pipe; then, we shall have

$$b=\tfrac{1}{4}(p+r)=\tfrac{1}{5}(b+p+r);$$

and the density after the first double stroke will be

$$d\frac{p+r}{b+p+r}=d\frac{4b}{5b}=\tfrac{4}{5}d.$$

Thus, by the first double stroke of the piston, one-fifth of the air contained in the receiver and pipe will be withdrawn, and the quantity remaining will be four-fifths of the original quantity. The density after the second stroke will, in like manner, be four-fifths of that after the first, or $\frac{16}{25}$ of the original density; and after the third, the density will be reduced to $\frac{64}{125}$, or nearly one-half. It thus appears that every three strokes will reduce the density nearly one-half; and consequently, that after twenty-seven strokes, the air would be reduced to about one-five-hundredth of its original density.

709. The preceding calculation is based upon the supposition that the relative capacities of the barrel, pipe, and receiver have been accurately ascertained, and that the mechanical construction of the pump is perfect, neither of which conditions is strictly fulfilled: and as it is frequently necessary to know the precise degree of exhaustion that has been attained, it becomes important to have a gauge, or index, by the aid of which we may ascertain the density of the remaining air at any moment. The instruments commonly employed for this purpose are,

1°. *The barometer gauge*, which consists of a straight glass tube about thirty-two inches in length, and open at both extremities. The tube is placed in a vertical position, its upper extremity communicating with the receiver of the pump, and its lower being immersed in a basin of mercury. When the process of exhaustion has been commenced, the air in the tube being rarefied, the pressure of the atmosphere upon the surface of the mercury in the basin will cause the mercury to rise in the tube, and the height at which it stands will indicate the difference between the exterior and interior pressures. These pressures are in the direct ratio of the densities of the air. The principal inconvenience of this gauge arises from the necessity of having a barometer with which to ascertain the pressure of the exterior air at the same time.

2°. *The short barometer gauge* is formed of a tube eight or ten inches in length, open at one extremity, and filled with mercury. This tube being inverted, and immersed at its open extremity in a basin of mercury, the pressure of the atmosphere upon the surface of the mercury in the basin will retain the tube entirely full. This apparatus being placed under a receiver which communicates with that of the pump, and the rarefaction being commenced, the short tube will remain full until the density of the air in the receiver has been so far reduced that its elastic force is insufficient to support a column of mercury of a length equal to that of the tube. The mercury in the tube will then fall, and its height at any moment will indicate the pressure of the air within. This gauge is evidently unfit for use when only a moderate degree of exhaustion is required.

3°. *The siphon gauge* is composed of a short bent tube, having two parallel branches, one of which is closed, and the other open. The closed branch being filled with mercury, and the tube being placed with the bend downwards, the mercury will be supported in that branch by the pressure of the exterior air. The tube is then placed beneath a receiver, and acts upon the same principle as the short barometer gauge, the bend in the tube serving as a substitute for the basin of mercury. This, also, is only applicable when a considerable degree of rarefaction is required.

710. The working of the piston being opposed by the pressure of the atmosphere on its superior surface, and this difficulty constantly increasing as the rarefaction proceeds, it has been found advantageous to adapt a second barrel to the pump, whose piston shall descend whilst that of the first barrel ascends,—and the reverse. The rods of the pistons have the form of a rack whose teeth engage in those of a wheel which is turned by a winch. The pressures on the pistons are thus caused to oppose each other, and the pump works with much greater ease. The rapidity of the exhaustion is likewise doubled by this arrangement.

711. If the construction of the pump be such as to require the lower valves to be opened by the elasticity of the air remaining in the receiver, the operation of the pump will evi-

dently cease whenever the rarefaction has been carried so far that the weight of the lower valve is sufficient to overcome the elastic force of the air within. To obviate this inconvenience, the lower valves are opened and closed by the motions of the piston, as shown in *Fig.* 245, which represents a sectional view of one of the most approved pumps. The disposition of the several parts has been somewhat altered, for the purpose of exhibiting them more clearly.

A represents the glass receiver resting upon the ground glass plate BC, and communicating by the cavity DFG with the two pump barrels VR and V'R'. The receiver likewise communicates by the cavity *svy* with the barometer gauge *yz*, immersed in the vessel of mercury M, and with the siphon gauge *vx*. E is a stopcock for cutting off the communication between the receiver and the barrels when the exhaustion has been effected, and E' a second stopcock for re-admitting the external air. In the best pumps, the barrels are made of glass, to prevent the corrosion which would take place by the action of the oil with which the pistons are lubricated to render them air-tight: for similar reasons, the pistons are sometimes made of steel. The racks L and L' of the pistons are worked by the wheel W, which is turned alternately to the right and left by the winch H. The lower valves V and V' are metallic, and have the form of a conic frustrum. To the back of the valve is attached a slender rod VR, which passes through an air-tight hole in the piston P, and carries near its upper extremity a small projection or shoulder. When the piston is raised, the friction of the valve-rod which passes through it causes the rod likewise to rise, opening the lower valve V: but this upward motion is soon checked by the shoulder coming into contact with the top of the barrel, and the rod then slides through the hole in the piston. Again, when the piston is depressed, it carries with it the valve-rod RV, closing the valve at the bottom of the pump, and the descent of the piston is then continued by sliding along the rod.

712. The valves of the pistons are variously constructed. In some instances they are metallic, resting upon a metallic bed; and in others, they are composed of strips of oiled silk,

bladder, or parchment, stretched across an opening in the piston, and alternately allowing and preventing the communication between the air beneath the piston and the exterior air. During the ascent of the piston, the valve remains closed by the stronger pressure of the atmosphere on its upper surface, and when the piston descends, the compressed air beneath it will force open the valve. This latter condition will always be fulfilled, whatever may be the degree of exhaustion, provided the piston can be forced into actual contact with the bottom of the barrel.

713. The pistons are usually composed of two metallic plates, which carry between them a packing of leather soaked in oil. The distance between these plates can be varied by means of a powerful screw; and by the application of a proper degree of pressure, the packing is caused to fit the barrel with accuracy.

714. By the aid of the air-pump we are enabled to exhibit many of the most important properties of atmospheric air:

1°. *The weight of the air* may be shown by screwing a vessel provided with a stopcock to the air-pump, and exhausting the air from within it. The weight of the vessel will be diminished by about $\frac{3}{10}$ of a grain for every cubic inch of air that has been withdrawn.

2°. *The pressure of the atmosphere* is rendered evident by the difficulty with which the receiver is removed from the plate of the pump after the air within it has been withdrawn.

A small strip of bladder being stretched across the mouth of an open receiver, and the air exhausted from beneath, the bladder will be ruptured by the pressure of the exterior air.

Two brass hemispheres, being ground so as to fit accurately to each other, and attached to the pump, cannot be separated without great difficulty after the air has been exhausted from the space enclosed by them. The pressure of the atmosphere is found to be equivalent to about 15 lb. for each square inch of surface exposed to its action.

3°. *The elasticity of the air* may likewise be shown by various experiments. If, for example, a bladder containing a small quantity of air be enclosed in a receiver, from which the air can be extracted, the elasticity of the air contained in the bladder will cause it to distend when the exterior pressure is removed; and on the re-admission of the air into the receiver, the bladder will again collapse.

If a light glass bulb, having an opening in its lower surface, be loaded with weights so that it will just sink in a vessel of water when the bulb is partially filled with water; upon withdrawing the air from the receiver in which the vessel of water has been deposited, the portion of air contained in the bulb will expand, expelling a portion of the water through the orifice in the bottom of the bulb. The bulb and weight will thus be rendered specifically lighter than water, and will consequently rise to the surface of the fluid in the vessel: upon re-admitting the air into the receiver, a portion of water will be forced into the bulb, and it will again sink.

4°. *The resistance of the air* to the motion of bodies may be exhibited by allowing two bodies of very unequal densities to fall in the exhausted receiver of the air-pump, and in the same receiver after the re-admission of the air. When the bodies fall in vacuo, they will reach the bottom of the receiver at the same instant; but when the receiver contains air, the denser body being least retarded by the resistance which the air offers, it will fall through the height of the receiver in much less time than that required by the rarer body.

Many other experiments may be contrived to illustrate the properties of air, but it is unnecessary to notice them in this place.

Of the Barometer.

715. The barometer is composed essentially of a bent tube ABC (*Fig.* 246), closed at A, and open at C, and filled with mercury throughout the portion NMBEF. The air is supposed to have been exhausted from the space AMN, and the column of mercury included between the planes MN and DFE is supported by the pressure of the atmosphere upon the surface FE. This column is usually about thirty inches in length, when the barometer is placed at the level of the ocean.

716. This instrument serves to indicate the changes which are constantly taking place in the pressure of the atmosphere; for, when the pressure becomes greater, the length of the column of mercury which it can sustain is necessarily increased, and the mercury therefore rises in the tube AD: but if, on the contrary, the pressure of the air should diminish, the length of the column will undergo a corresponding diminution.

The pressure of the atmosphere at any point being that due to the weight of a column of air extending from that point to the top of the atmosphere, it follows that this pressure will decrease as we ascend above the earth's surface, and consequently, that the height of the mercurial column will diminish.

717. This principle has been employed to determine the difference of level of two places situated at unequal distances above the surface of the earth. For the purpose of investigating a formula which shall be applicable to this object, we shall denote by

h' . . . . the height of the mercurial column at the lower station,
h . . . . the height of the mercurial column at the upper station,
D' and D the corresponding densities of the atmosphere at the two stations.

Then, if we suppose the axis of z to be vertical, the general equation of equilibrium of heavy fluids as obtained in Art. 655, will be

$$dp = Dgdz.^*$$

Let the origin be assumed at the lower station, and let the co-ordinates z be reckoned positive upwards; then, as we ascend in the atmosphere, the pressure arising from the weight of the superincumbent strata will diminish, and the

---

[*] This result may be obtained directly by considering a column of the atmosphere, whose base AB (*Fig.* 247) is the unit of surface: the pressure sustained by this base is measured by the weight of the column of air ABDC extending to the top of the atmosphere; and the elementary pressure dp will be represented by the weight of a column having the same base, and a height equal to dz. The base of this elementary column being equal to unity, its volume will be expressed by $1 \times dz$, or dz, and its mass by Ddz: thus, $gDdz$ represents the weight which will measure the elementary pressure dp. This result will obviously be independent of the particular form given to the base AB which has been assumed as the superficial unit.

density of the air will undergo a corresponding decrease. Thus, the pressure p being a decreasing function of the altitude z, dp and dz will be affected with contrary signs: hence, the preceding equation should be written

$$dp=-\mathrm{D}g\,dz \;\ldots\ldots\; (412).$$

If the difference of level of the two places be but slight, the force of gravity g may be regarded as constant; and hence we shall obtain, by integration,

$$z=-\frac{1}{g}\int\frac{dp}{\mathrm{D}} \;\ldots\ldots\; (413).$$

But it has been shown (Arts. 651 and 696) that when the temperature is supposed constant, the pressure and density are proportional to each other; hence, if P denote the pressure capable of producing a density represented by unity, we shall have

$$p=\mathrm{PD};$$

and therefore,

$$dp=\mathrm{P}d\mathrm{D}:$$

this value substituted in equation (413) gives

$$z=-\frac{\mathrm{P}}{g}\int\frac{d\mathrm{D}}{\mathrm{D}};$$

and by effecting the integration indicated, there results

$$z=-\frac{\mathrm{P}}{g}\log \mathrm{D}+\mathrm{C}.$$

To determine the constant, we remark, that when $z=0$, the density becomes that which we have supposed to exist at the lower station, and which has been denoted by D'. Thus, the preceding equation becomes

$$0=-\frac{\mathrm{P}}{g}\log \mathrm{D}'+\mathrm{C};$$

eliminating C between this equation and the preceding, we find

$$z=\frac{\mathrm{P}}{g}(\log \mathrm{D}'-\log \mathrm{D}),$$

or,

$$z=\frac{\mathrm{P}}{g}\log\frac{\mathrm{D}'}{\mathrm{D}}.$$

But the densities being proportional to the pressures, they will likewise be proportional to the observed altitudes of the mercurial column: hence,

$$h : h' :: D : D', \text{ or } \frac{h'}{h}=\frac{D'}{D};$$

this value of $\frac{D'}{D}$ being substituted in that of z, we obtain

$$z=\frac{P}{g}\log\frac{h'}{h}.$$

The logarithm of $\frac{h'}{h}$, which appears in this expression, appertains to the Naperian system: if therefore, we represent by $\text{Log}\,\frac{h'}{h}$, the tabular logarithm of $\frac{h'}{h}$, and by M the reciprocal of the modulus, we shall have

$$M\,\text{Log}\,\frac{h'}{h}=\log\frac{h'}{h};$$

and, by substitution,

$$z=\frac{MP}{g}\,\text{Log}\,\frac{h'}{h} \,\ldots\ldots\, (414).$$

718. To determine the value of the constant P, which represents the pressure exerted upon the unit of surface, and capable of producing a density of air represented by unity, we remark, that the density D' at the lower station corresponds to the pressure exerted by the atmosphere at that point: this pressure is measured by the weight of a column of air whose base is the superficial unit, and whose altitude is equal to that of the atmosphere: but this column of air is equal in weight to the mercurial column whose height is h'; if therefore D'' denote the density of mercury, the mass of the column will be expressed by $1 \times h'D''$, or $h'D''$: and by multiplying this product by g, we shall obtain the expression $h'D''g$, for the weight of the column supported at the lower station. Such will be the pressure capable of producing the density D'. To obtain the pressure P corresponding to the unit of density, we make the proportion

$$D' : 1 :: h'D''g : P;$$

whence,

$$P=\frac{h'D''g}{D'};$$

substituting this value in the formula (414), there results

$$z=\frac{\mathrm{M}h'\mathrm{D}''}{\mathrm{D}'}\,\mathrm{Log}\,\frac{h'}{h}\ \ldots\ldots\ (415).$$

719. The intensity of the force of gravity being different at different places on the surface of the earth, the weight of the same column of mercury will likewise vary when it is transported from one place to another: thus, if the force of gravity be denoted by g at one station, and by $(1-\delta)g$ at a second, the mercurial column whose height is h' will become heavier or lighter at the second station than it was at the first, according as δ is negative or positive.

Let the quantity δ be considered positive: then $1-\delta$ will be positive, and less than unity, since the variations of gravity are exceedingly small. But a column of mercury whose height is h' becoming lighter at the point whose gravity is denoted by $(1-\delta)g$, it will correspond to a less pressure of the atmosphere, and hence, the density of the air corresponding to this pressure will be less.

The densities of the air being proportional to the pressures exerted, and these pressures being measured by the weights of the column of mercury whose height is h', it follows that the intensities of gravity, which are represented respectively by g and $(1-\delta)g$ at the two places, will be proportional to the densities corresponding to the same height h' of the mercurial column: thus, if we denote by d the density of the air at the place where the intensity of gravity is represented by $(1-\delta)g$, we shall have

$$g : g(1-\delta) :: \mathrm{D}' : d;$$

whence,

$$d=\mathrm{D}'(1-\delta).$$

This value of the density must be substituted in the formula (415), in order that it may become applicable to the place at which the gravity is represented by $(1-\delta)g$: the formula will thus become

$$z=\frac{\mathrm{D}''\mathrm{M}h'}{\mathrm{D}'(1-\delta)}\,\mathrm{Log}\,\frac{h'}{h}\ \ldots\ldots\ (416).$$

From a comparison of the results obtained by causing pendulums to oscillate in different latitudes, it has been ascertained that if the intensity of gravity be denoted by g at the latitude of 45°, the quantity δ will be expressed by $0.002837 \times \cos 2\psi$, when the latitude is supposed to become equal to ψ. Hence, by substitution in the preceding formula, we obtain

$$z = \frac{D'Mh' \operatorname{Log} \frac{h'}{h}}{D'(1 - 0.002837 \cos 2\psi)}.$$

The quantity δ being always extremely small, we may replace $\frac{1}{1-\delta}$ in equation (416) by its development $1+\delta+\delta^2+\delta^3+$&c. and neglect the terms δ^2, δ^3, &c. as extremely minute with reference to δ: we thus obtain $\frac{1}{1-\delta}=1+\delta$; hence, the value of z will become

$$z = \frac{D''Mh'}{D'}(1+0.002837 \cos 2\psi) \operatorname{Log} \frac{h'}{h} \ldots\ldots (417).$$

720. This formula has been obtained upon the supposition that the temperature remains constant in passing from the lower to the higher station. To adapt the formula to the case in which the temperature is variable, it will be necessary to know the law according to which air expands when subjected to a change of temperature. The experiments of Gay Lussac and other philosophers demonstrate conclusively that atmospheric air when perfectly dry, and when subjected to a constant pressure, expands for each degree of Fahrenheit's thermometer, between the temperatures of 32° and 212°, $\frac{1}{480}$ of its volume at the temperature of 32°. Thus a volume of air represented by unity at the temperature of 32°, will become $1+\frac{n}{480}$ when its temperature has been raised to $32°+n°$: and since the densities are in the inverse ratio of the spaces occupied by the same mass, it follows that the density d' of the air, at the temperature $32°+n°$, will be expressed by

$$d' = \frac{D'}{1+\frac{n}{480}},$$

D′ being the density at the temperature 32°.

The coefficient of the number n being very small, the error which will be introduced by assigning to n a value which shall not differ greatly from its real value will always be extremely small; and since the variations in temperature which occur in passing from a lower to a higher station are nearly uniform, we may, without sensible error, regard the temperature as constant, provided we assign to it a value equal to the arithmetical mean between the temperatures t and t' at the higher and lower stations; we shall thus have

$$n=\frac{t+t'}{2}-32;$$

and the density d' of the air, which was previously represented by D′, will become

$$d'=\frac{D'}{1+\frac{1}{480}\left(\frac{t+t'}{2}-32\right)}=\frac{D'}{1+\frac{t+t'-64}{960}} \dots\dots (418).$$

But the density of mercury being increased by a diminution in the temperature, the height of the mercurial column at the colder station will be less, for the same pressure, than it would have been if the temperature had remained constant, and equal to that at the warmer station; and since mercury is found to expand about $\frac{1}{9742}$ part of its bulk for every degree of Fahrenheit's thermometer, it will be necessary to increase the height h, which is supposed to correspond to the colder station, by the quantity $\frac{h}{9742}$ taken as many times as there are degrees of difference between the temperatures of the mercury at the two stations, in order to reduce this height h to what it would have been, if the temperature had remained constant, and equal to that at the warmer station. Let T and T′ represent the temperatures of the mercury at the two stations as indicated by thermometers in contact with the barometers; then the quantity h in equation (417) should be replaced by

$$h+\frac{h(T'-T)}{9742};$$

hence, by substituting this value for h, and that of d' (418) for D′, in formula (417), we find

$$z=\frac{D'Mh'}{D'}\left(1+\frac{t+t'-64}{960}\right)$$

$$\times(1+0.002837\cos 2\psi)\,\mathrm{Log}\,\frac{h'}{h\left(1+\frac{T'-T}{9742}\right)}.$$

721. Let it be supposed that the observations which determine the height h' are made in the latitude of 45°, and at the level of the ocean; we shall have

$$\cos 2\psi=0;$$

and the preceding formula will give

$$\frac{D'Mh'}{D'}=\frac{z}{\left(1+\frac{t+t'-64}{960}\right)\mathrm{Log}\,\frac{h'}{h\left(1+\frac{T'-T}{9742}\right)}}\;\ldots\ldots(419).$$

If the height z be measured trigonometrically, and the quantities h, h', t, t', T, T' be determined by taking a mean result of a great number of observations, the second member of this equation will become entirely known, and therefore the constant $\frac{Mh'D'}{D'}$ will likewise be known. This constant has been thus found to be equal to 60345 feet: if its value be substituted in that of z, we shall obtain the following formula:

$$z=60345\text{ ft.}\left(1+\frac{t+t'-64}{960}\right)(1+0.002837\cos 2\psi)$$

$$\times\mathrm{Log}\,\frac{h'}{h\left(1+\frac{T'-T}{9742}\right)}\;\ldots\ldots(419\,a).$$

722. The second member of this equation may be put under a more convenient form; for, we have

$$\frac{t+t'-64}{960}=.001042(t+t'-64);$$

and if we denote by θ the difference between the temperatures T and T', and change the form of the last factor in equation (419 a), there will result

$$\mathrm{Log}\,\frac{h'}{h\left(1+\frac{T'-T}{9742}\right)}=\mathrm{Log}\,h'-\mathrm{Log}\,h-\mathrm{Log}\left(1+\frac{\theta}{9742}\right);$$

or, by developing the last term of the second member, retaining only the first term of the development, we shall have

$$\text{Log}\,\frac{h'}{h\left(1+\frac{T'-T}{9742}\right)}=\text{Log}\,h'-\text{Log}\,h-\frac{1}{9742}M't\,;$$

in which M′ represents the modulus of the system. The numerical value of the coefficient of t is .000044. Hence, the equation (419 *a*) may be reduced to

$$z=60345\text{ ft. }[1+.001042(t+t'-64)](1+.002837\cos 2\psi)$$
$$\times(\text{Log}\,h'-\text{Log}\,h-.000044t).$$

723. To apply this formula to the determination of the difference of level of two stations, it will be necessary to observe carefully the altitude of the mercurial columns at each station, and the temperature of the atmosphere as indicated by a thermometer placed in the shade, and at some distance from the barometer. The temperature of the mercury as shown by a thermometer in contact with the tube of the barometer should likewise be noted. These observations should be made at the same instant, by different observers, at the two stations, in order to avoid the errors which might arise from a change of pressure or temperature during the interval between the observations. When the condition of simultaneous observations becomes impracticable, it will be advisable to make observations at one of the stations, the lower for example, at equal intervals before and after the time of observation at the other station. Then, an arithmetical mean between the first and last results may be considered as nearly equivalent to an observation made at the instant corresponding to the mean between the two times, provided the interval be but short, and the difference between the results of the two observations inconsiderable.

724. The general formula for the difference of level of two stations having been obtained upon the supposition that the atmosphere is in equilibrio, the results given by it are to be relied on most confidently when the observations have been made in calm weather.

PART FOURTH.

HYDRODYNAMICS.

OF THE DISCHARGE OF FLUIDS THROUGH HORIZONTAL ORIFICES.

725. EXPERIENCE has shown, that when a fluid issues from a small orifice in the bottom of a vessel, the superior surface of the fluid maintains itself in a position sensibly horizontal, during the discharge of the fluid. Hence, if we conceive the fluid divided into horizontal strata, these strata may be regarded as preserving their parallelism during their descent, and the particles may be considered as descending in vertical lines. This hypothesis, however, can only be regarded as approximating to the truth; for, if the form of the vessel be not prismatic, it will be impossible for any one stratum to occupy the place of that immediately beneath it without undergoing some change in its dimensions; its particles will therefore be subjected to horizontal motions. Moreover, the particles situated in the immediate vicinity of the orifice, being without support, yield to the pressure exerted against them by the adjacent particles, and thereby tend to deflect the latter from their vertical directions: but, in what follows, we shall omit the consideration of these circumstances, which would greatly complicate the question, and which are found to produce but a slight effect when the form of the vessel is nearly prismatic. The accuracy of the hypothesis may be rendered evident by mixing with the fluid an insoluble powder of nearly the same density: the particles of this powder will be carried along with the fluid, and the paths which they describe may be readily observed. In this manner it will be found that the particles descend nearly in vertical lines until they approach very near to the orifice.

726. Let it be supposed that the ordinate z measures the distance *mo* (*Fig.* 248) of one of the fluid strata from a horizontal plane AB, which will be assumed as coinciding with the surface of the fluid. The form of the vessel being determined by the equation of its interior surface, $f(x, y, z,)=0$, we can deduce from this equation the area s of the section which corresponds to the ordinate z, and by multiplying this section by dz, the thickness of a stratum, we shall obtain sdz for the volume of the stratum. This being premised, it is obvious that all the particles composing a single stratum will have a common velocity; but the particles of different strata will have different velocities; for, the fluid being supposed incompressible, any one stratum in descending through the height dz, in the time dt, will cause a volume of fluid equal in volume to the stratum to issue through the orifice. But if we denote by u the velocity of the fluid at the orifice EF, and by k the area of the orifice, the space described by a particle issuing from the orifice, in the time dt, will be expressed by udt, and the quantity of fluid discharged in the same time will be represented by $kudt$. Equating this value with that of the stratum, we shall obtain

$$sdz=kudt \ldots\ldots (420);$$

whence,

$$ku=s\frac{dz}{dt} \ldots\ldots (421).$$

727. At the expiration of the time t, the velocity of the stratum whose section is s will be equal to $\frac{dz}{dt}$, and if this velocity be represented by v, the equation (421) will reduce to

$$ku=sv \ldots\ldots (422);$$

whence we conclude, that the velocities v and u are in the inverse ratio of the sections s and k. This result might have been anticipated; for, the velocity must evidently increase in the same ratio that the area of the section is diminished, in order that the quantity of fluid passed through the section may remain constant.

728. At the expiration of the time $t+dt$, the velocity v will become $v+\frac{dv}{dt}dt$: but if the motions of the particles were

independent of their action upon each other, the incessant force g, which solicits them, would communicate, in the instant dt, the velocity gdt; hence, the velocity lost by the stratum whose section is s, and velocity v, in the time dt, will be expressed by $gdt-\frac{dv}{dt}dt$, and consequently, the incessant force due to this velocity will be represented by

$$g-\frac{dv}{dt}.$$

But by the principle of D'Alembert, an equilibrium would subsist in the system if each fluid stratum were acted upon by the force lost $\left(g-\frac{dv}{dt}\right)$ which corresponds to it. This supposition will convert the equation $dp=\mathrm{D}gdz$ (Art. 655) into

$$dp=\mathrm{D}\left(g-\frac{dv}{dt}\right)dz \ldots\ldots (423).$$

The quantity dp represents the differential of the pressure at that stratum of the fluid which corresponds to the ordinate z, whilst the fluid is in motion. For, the force g, which acts on each stratum, being resolved into two components, one of which $\frac{dv}{dt}$ is just capable of producing the motion assumed by the stratum, the other component $g-\frac{dv}{dt}$ will obviously be alone effective in producing a pressure on the other strata; and the expression for dp has been obtained upon the supposition that these second components were alone applied to the fluid particles.

The differential dv which enters into the preceding equation must be replaced by its value deduced from formula (422); from that formula we obtain

$$v=\frac{ku}{s} \ldots\ldots (424).$$

729. The second member of this equation contains the two variables u and s: the quantity u, which expresses the velocity at the orifice, is a function of the time, and the section s is a function of the ordinate z.

The differential of v regarded as a function of z expresses the difference between the velocities of two consecutive sections, these velocities being considered at the same instant. But if the differential be taken with reference to t as a variable, we shall obtain the difference between the velocities of two consecutive strata which pass in succession through the same section of the vessel. And, lastly, if we wish to obtain the difference between the consecutive velocities of the same stratum, we must differentiate v with reference to the two variables t and z, regarding the latter as a function of the former.

This last supposition should be adopted in finding the value of dv as employed in equation (423). We shall therefore differentiate the second member of (424), regarding u as a function of t, and s as a function of z, which is itself a function of t. But the differential of (424) being in general

$$dv=\frac{k}{s}du+kud\frac{1}{s},$$

or,

$$dv=\frac{kdu}{s}-ku\,.\,\frac{ds}{s^2},$$

it will become, when modified according to the hypothesis assumed,

$$dv=\frac{k}{s}\times\frac{du}{dt}dt-\frac{ku}{s^2}\times\frac{ds}{dz}\times\frac{dz}{dt}dt.$$

If we deduce the value of $\frac{dv}{dt}$ from this expression, and substitute it in equation (423), we shall obtain

$$dp=\mathrm{D}\left(gdz-\frac{k}{s}\,.\,\frac{du}{dt}dz+\frac{ku}{s^2}\,.\,\frac{ds}{dz}\,.\,\frac{dz}{dt}dz\right):$$

eliminating $\frac{dz}{dt}$ by means of equation (420), there results

$$dp=\mathrm{D}\left(gdz-k\frac{du}{dt}\,.\,\frac{dz}{s}+k^2u^2\frac{ds}{s^3}\right).$$

730. This equation must be integrated with reference to z. We remark, however, that s will necessarily vary with z, but that the quantities u and $\frac{du}{dt}$, which represent the particular

values of v and $\frac{dv}{dt}$ corresponding to the orifice, not being functions of the quantity z, they may be regarded as constant in effecting this integration.

731. If we regard u and $\frac{du}{dt}$ as constant, it is obvious that all the integrals will be taken with reference to z, and therefore apply merely to the dimensions of the vessel. But, when these integrals have been obtained, we may regard u and $\frac{du}{dt}$ as variables, and functions of t.

732. By effecting the integration, we obtain

$$p=\mathrm{D}\left(gz-k\frac{du}{dt}\int\frac{dz}{s}-\frac{k^2u^2}{2s^2}\right)+\mathrm{C}\,\ldots\ldots\,(425).$$

The velocity u which enters into this equation is equal to the value $\frac{dz}{dt}$ corresponding to the orifice, and will obviously be a function of the time. Consequently, as the quantity u has been supposed constant in the preceding integration, the time t must be constant likewise. Hence, the constant C will in general be a function of the time.

733. To determine this constant, let P represent the pressure sustained by the superior surface CD of the fluid (*Fig.* 249), the area of this surface being denoted by s'. If the integral $\int\frac{dz}{s}$ be taken in such manner that it shall be equal to zero when s becomes equal to s', this section s' will correspond to an ordinate $z'=\mathrm{OL}$, and the equation (425) will give, upon this hypothesis,

$$\mathrm{C}=\mathrm{P}-\mathrm{D}\left(gz'-\frac{k^2u^2}{2s'^2}\right).$$

This value being substituted in (425), we obtain

$$p=\mathrm{P}+\mathrm{D}\left[g(z-z')-k\frac{du}{dt}\int\frac{dz}{s}+\tfrac{1}{2}u^2\left(\frac{k^2}{s'^2}-\frac{k^2}{s^2}\right)\right]\ldots\,(426).$$

734. This pressure is exerted at every point of the stratum whose distance from the plane AB is equal to z. If we wish to obtain the pressure Q at the orifice, we denote by z'' the

corresponding value of the ordinate z which will be equal to On, and observe that the section s will, at that point, be equal to k: the integral $\int\frac{dz}{s}$ being then taken between the limits $z=z'$ and $z=z''$, we shall obtain, by representing this integral by N, and substituting these values in equation (426),

$$Q-P=D\left[g(z''-z')-kN\frac{du}{dt}+\tfrac{1}{2}u^2\left(\frac{k^2}{s'^2}-1\right)\right].$$

735. This equation makes known the pressure at the orifice: the first member expresses the difference between the pressures at the orifice and at the surface. Let these pressures be supposed equal, as is the case when they arise from the weight of the atmosphere: then, Q−P will reduce to zero, the common factor D will disappear, and there will remain

$$g(z''-z')-kN\frac{du}{dt}+\tfrac{1}{2}u^2\left(\frac{k^2}{s'^2}-1\right)=0;$$

but the area k of the orifice being always supposed less than the area s' of the superior surface, the fraction $\frac{k^2}{s'^2}$ will be less than unity; if therefore, we wish to render the coefficient of u^2 positive, we may write this equation under the form

$$g(z''-z')-kN\frac{du}{dt}-\tfrac{1}{2}u^2\left(1-\frac{k^2}{s'^2}\right)=0 \;.\;.\;.\;.\;.\; (427).$$

736. If in this equation we introduce the vertical distance of the orifice below the surface of the fluid, making

$$z''-z'=h \;.\;.\;.\;.\;.\; (428),$$

we shall have

$$gh-kN\frac{du}{dt}-\tfrac{1}{2}u^2\left(1-\frac{k^2}{s'^2}\right)=0 \;.\;.\;.\;.\;.\; (429).$$

The quantity h, which represents the distance EP (*Fig.* 250), will be constant if the surface of the fluid be supposed to be maintained at the same height; but it will be variable if the vessel be supposed to discharge its contents without being replenished.

737. In the latter case, if we make EO$=a$, PO$=z$, and EP$=h$, we shall have the relation

$$h=a-z \;.\;.\;.\;.\;.\; (430);$$

and the equation (429) will become

$$g(a-z)-k\text{N}\frac{du}{dt}-\tfrac{1}{2}u^2\left(1-\frac{k^2}{s'^2}\right)=0\ \ldots\ldots\ (431).$$

738. If the surface of the fluid be constantly maintained at the same height, the quantity h will have a constant value, and the integral N, which will then be a function of constant quantities, will likewise be invariable. Thus, equation (429), containing no other variables than t and u, may be put under the form

$$a-b\frac{du}{dt}-cu^2=0\,;$$

from which we deduce

$$dt=\frac{bdu}{a-cu^2}.$$

This equation can be readily integrated by the method of rational fractions; for, if we make $b=b'c$, and $a=a'^2c$, the quantity c will become a factor of the numerator and denominator, and may be stricken out; whence we obtain

$$dt=\frac{b'du}{a'^2-u^2}.$$

The second member of this equation being resolved into factors, we shall have

$$dt=\frac{\frac{b'}{2a'}du}{a'+u}+\frac{\frac{b'}{2a'}du}{a'-u},$$

which, being integrated, gives

$$t=\frac{b'}{2a'}\log(a'+u)-\frac{b'}{2a'}\log(a'-u)+\text{C}\,;$$

or,

$$t=\frac{b'}{2a'}\log\frac{a'+u}{a'-u}+\text{C}.$$

The constant C is determined by the condition that the velocity u is equal to zero at the same instant as the time t; thus, the supposition of $u=0$, and $t=0$, reduces the preceding equation to

$$\frac{b'}{2a'}\log 1+\text{C}=0\,;$$

or,

$$C=0:$$

whence,

$$t=\frac{b'}{2a'}\log\frac{a'+u}{a'-u};$$

this equation will determine u, if we suppose the time t to be given.

739. If we denote by e the base of the Naperian system, and pass from logarithms to numbers, we shall obtain

$$\frac{a'+u}{a'-u}=e^{\frac{2a't}{b'}};$$

and by resolving the equation with reference to u, there results

$$u=\frac{a'\left(e^{\frac{2a't}{b'}}-1\right)}{\left(e^{\frac{2a't}{b'}}+1\right)};$$

or replacing a' and b' by their values (Arts. 737 and 738), the expression for the velocity will become

$$u=\sqrt{\left(\frac{2gh}{1-\frac{k^2}{s'^2}}\right)}\times\frac{e^{\frac{t}{kN}\sqrt{\left[2gh\left(1-\frac{k^2}{s'^2}\right)\right]}}-1}{e^{\frac{t}{kN}\sqrt{\left[2gh\left(1-\frac{k^2}{s'^2}\right)\right]}}+1}.$$

But, if the area of the orifice, which is denoted by k, be supposed extremely small, the exponent of e, increasing with the time t, will become exceedingly great after the expiration of a very short time. Hence, we may neglect unity in the numerator and denominator of the last factor, as very small with reference to the term which precedes it, and the value of u will then be reduced to

$$u=\sqrt{\left(\frac{2gh}{1-\frac{k^2}{s'^2}}\right)}=2gh,$$

by neglecting k^2 with reference to s'^2.

Thus, it appears that the expression $\sqrt{(2gh)}$ is a limit which the velocity of the fluid at the orifice never attains, but to which this velocity becomes very nearly equal after the expiration of an exceedingly short time.

The value of the velocity being thus determined, we substitute it in equation (425), and thence deduce the pressure on the unit of surface.

740. If the vessel be supposed to empty itself, the upper surface will be depressed as the fluid is discharged, and the quantity h, or $(a-z)$ must therefore be regarded as variable in equations (429) and (431).

The equation (431) will thus contain the three variables t, u, and z, and will consequently be insufficient for the solution of the problem: but a second relation may be obtained by means of equation (420) in which we replace s by s', and thus obtain

$$ku=s'\frac{dz}{dt} \quad \ldots\ldots (432).$$

741. This equation likewise contains three variables, and we are therefore unable to integrate it; but it will serve to eliminate z. For this purpose, we differentiate equation (431), which gives

$$-g\frac{dz}{dt}-k\mathrm{N}\frac{d^2u}{dt^2}-u\frac{du}{dt}\left(1-\frac{k^2}{s'^2}\right)=0,$$

and by eliminating $\frac{dz}{dt}$, we obtain

$$-g\frac{ku}{s'}-k\mathrm{N}\frac{d^2u}{dt^2}-u\frac{du}{dt}\left(1-\frac{k^2}{s'^2}\right)=0.$$

This equation, which can only be integrated by approximation, makes known the relation between the time and the velocity.

742. When the orifice is supposed extremely small, the terms containing k may be neglected, and the equation (426) will be reduced to

$$p=\mathrm{P}+\mathrm{D}g(z-z');$$

but $z-z'$ is represented by $On-OL$ (*Fig.* 249); and it will therefore express the distance of the point n, whose ordinate is equal to z, beneath the surface of the fluid. Hence, the pressure p exerted upon the unit of surface at the point n is equal to the pressure P at the surface of the fluid, plus the pressure arising from a column of a fluid whose height is equal to the distance of this point below the surface.

It should be remarked, that this pressure is precisely that which would be exerted at the point n if the fluid were supposed at rest.

743. If the terms containing k in equation (429) be neglected as infinitely small, it will reduce to

$$gh-\tfrac{1}{2}u^2=0;$$

whence,

$$u=\sqrt{(2gh)} \ldots\ldots (433):$$

and we therefore conclude, that when a fluid escapes from an infinitely small orifice in the bottom of a vessel, the velocity will be the same as that acquired by a heavy body in falling through a distance equal to the height of the surface of the fluid in the reservoir above the orifice; and since it has been shown (Art. 405) that a body projected vertically upwards will rise to a height equal to that through which it must fall to acquire the velocity of projection, it follows, that if by means of a curved tube, the jet of fluid be directed upwards, it will rise to the level of the surface of the fluid in the reservoir.

744. The expression for the velocity with which a fluid will issue from an extremely small orifice in the bottom of a vessel may be investigated in a more elementary manner, as follows. Let EF (*Fig.* 251) represent a very small orifice in the bottom of a vessel ABCD, which is filled with a fluid to the level AB, and let GF represent an infinitely thin stratum of the fluid directly above the orifice EF. Denote the height of this stratum by dh, the entire height of the fluid FI being represented by h. Then if the stratum of fluid GF be supposed to fall through the height HF under the influence of the force of gravity, it will acquire a velocity v, expressed by

$$v=\sqrt{(2g\times\mathrm{FH})}=\sqrt{(2g\times dh)}.$$

But if the stratum be supposed to descend through the same height, being urged by its weight and the pressure arising from the column of fluid GI, which is directly over it, the incessant force g', which is then exerted upon it, will be to the force of gravity, as FI to FH. Hence, we shall have

$$\frac{g'}{g}=\frac{\mathrm{FI}}{\mathrm{FH}}=\frac{h}{dh}.$$

Again, if v' denote the velocity acquired by the stratum in descending through the space FH, when urged by the force g', we shall have

$$v'=\sqrt{(2g'\times \mathrm{FH})}=\sqrt{(2g'\times dh)};$$

and by comparing this value with that of v, we find

$$\frac{v'}{v}=\frac{\sqrt{(2g'\times dh)}}{\sqrt{(2g\times dh)}};$$

or, by substituting the value of $\frac{g'}{g}$, and reducing, there results

$$v'=v\sqrt{\frac{h}{dh}}=\frac{v}{\sqrt{dh}}\sqrt{h}=\sqrt{(2gh)}.$$

This expression is precisely the same as that which would be obtained for the velocity of a body falling freely through the height FI.

745. When the orifice, which is still supposed exceedingly small, is pierced in the vertical face of a vessel, the fluid will issue in a horizontal direction, and will describe the arc of a parabola, if the resistance of the air be neglected. The angle of projection denoted by α in equation (289), being in the present case equal to zero, we shall have tang $\alpha=0$, cos $\alpha=1$: these suppositions reduce the formula (289) to

$$4hy=x^2;$$

an equation of a parabola whose axis is vertical, and whose vertex coincides with the origin of co-ordinates.

746. The distance to which the fluid will spout upon a horizontal plane situated at any distance below the orifice may be readily determined. For, let O (*Fig.* 252) represent an orifice in the vertical side of a vessel which is filled with a fluid to the level EF; and let AB represent the horizontal plane upon which the jet is allowed to fall. Then, the quantity h will represent the distance OF, and the ordinate CD of the parabola OD will be determined by making $y=\mathrm{OC}$: we thus obtain

$$\mathrm{CD}=x=\sqrt{(4hy)}=2\sqrt{(\mathrm{OF}\times \mathrm{OC})}.$$

But the expression $\sqrt{(\mathrm{OF}\times \mathrm{OC})}$ is equal to the ordinate OG of a semicircle described upon CF as a diameter. Hence, we derive the following rule: *The horizontal distance to which a fluid will spout from an orifice in the vertical side of* *a vessel, is equal to double the ordinate of a semicircle described upon the distance intercepted between the upper surface of the fluid and the horizontal plane upon which the fluid falls; this ordinate being drawn through the point which corresponds to the orifice.*

When the orifice is pierced at the middle of the line CF, the ordinate OG will be a maximum, and the distance to which the fluid will spout will therefore be the greatest.

747. The velocity u having been determined, we can readily ascertain the quantity of fluid discharged in the time t. For this purpose, we remark, that whilst the stratum of fluid CD (*Fig.* 250) sinks to the level MN, a volume of fluid equal to that contained between the planes CD and MN must pass through the orifice. But if we represent by s a section of the vessel, and by dz the thickness of an elementary stratum, the integral $\int sdz$ taken between limits CD and MN will express the volume of fluid discharged. If this volume be denoted by Q, we shall have

$$Q=\int sdz \; \ldots\ldots \; (434):$$

but the equation (420) gives

$$sdz=kudt;$$

whence, by substitution, we obtain

$$Q=\int kudt.$$

The value of the quantity discharged may be deduced immediately from that of the velocity. For, if de represent the space passed over by the fluid filament in the time dt, upon leaving the orifice, we shall have

$$udt=de:$$

and if this expression be multiplied by k, the area of the orifice, we shall obtain $kudt$ for the volume discharged in the time dt. Taking the integral $\int kudt$, we shall find the quantity discharged in the time t.

To effect the integration, we replace u by its value $\sqrt{(2gh)}$ given in equation (433): we thus find

$$Q=k\sqrt{(2g)}\int\sqrt{h}\,.\,dt \; \ldots\ldots \; (435).$$

748. Two distinct cases may now be presented, viz. when h is constant, and when h is variable. The first occurs

when the fluid in the reservoir is constantly maintained at the same height, and the preceding equation can then be integrated without difficulty, since the quantity h may be replaced by a constant a.

Thus, we shall have

$$Q=kt\sqrt{(2ga)}+C.$$

The constant C may be determined by the condition that the quantity Q is equal to zero at the commencement of the time, or $Q=0$, and $t=0$; hence,

$$C=0;$$

and the equation therefore reduces to

$$Q=kt\sqrt{(2ga)} \text{ . . . . . (436).}$$

749. If the orifice k be supposed circular, its radius being represented by r, we shall have

$$k=\pi r^2;$$

and the formula will become

$$Q=\pi\sqrt{(2g)}tr^2\sqrt{a} \text{ . . . . . (437).}$$

The quantity $\pi\sqrt{(2g)}$ will be the same for all problems which may be proposed, and its value may be immediately deduced, since we have

$$\pi=3.14159, \quad g=32.1598.$$

The quantity g being expressed in feet, the values of r and a must be expressed in units of the same kind, and the quantity discharged will then be expressed in cubic feet.

750. The time t must be expressed in seconds, since the second has been adopted as the unit of time in determining the value of g.

751. If the fluid be water, the weight of the quantity discharged may be determined by allowing 62½lbs for every cubic foot.

752. The formula (437) likewise serves to determine the time necessary for a given quantity of fluid to be discharged from an orifice in a vessel, when the fluid is maintained at a constant height; for the formula gives

$$t=\frac{Q}{\pi\sqrt{(2g)}r^2\sqrt{a}} \text{ . . . . . (438).}$$

753. As an example, let the vessel be supposed cylindrical, the radius of its base being denoted by b; and let it be required to determine the time necessary to discharge a volume of fluid equal to that of the cylinder.

In this case, the horizontal sections being all equal to πb^2, the equation (434) will give

$$Q=\int \pi b^2\, dz\,;$$

and consequently,

$$Q=\pi b^2 z+C.$$

Taking the integral between the limits $z=0$ and $z=a$, there results

$$Q=\pi b^2 a.$$

This value substituted in formula (438) gives

$$t=\frac{\pi a b^2}{\pi\sqrt{(2g)}r^2\sqrt{a}}\,;$$

or, by reduction,

$$t=\frac{b^2\sqrt{a}}{r^2\sqrt{(2g)}}.$$

754. If we suppose the fluid to be maintained at a height a' in a second vessel, and denote by Q' the quantity discharged from an orifice k' in the time t, the equation (436), when applied to the present case, will give

$$Q'=k'\sqrt{(2g)}.\,t\sqrt{a'}\,;$$

and by comparing this equation with (436), we can establish the proportion

$$Q : Q' :: k\sqrt{(2g)}.\,t\sqrt{a} : k'\sqrt{(2g)}.\,t\sqrt{a'}\,;$$

or, by suppressing the common factor $t\sqrt{(2g)}$, this proportion becomes

$$Q : Q' :: k\sqrt{a} : k'\sqrt{a'}.$$

Hence it appears, that *the quantities discharged in the same time, from orifices of different sizes, and situated at different depths, are directly proportional to the areas of those orifices and the square roots of their depths.*

755. From the formula (436) we can deduce another convenient theorem relative to the quantity of fluid discharged. For, let s represent the space through which a body would fall in the time t; we shall have

$$s=\tfrac{1}{2}gt^2,$$

or,

$$t=\sqrt{\frac{2s}{g}}.$$

Substituting this value for t in equation (436), we obtain

$$Q=2k\sqrt{(as)};$$

and since $\sqrt{(as)}$ is equal to a mean proportional between the distances a and s, we deduce the following rule: *The volume of fluid discharged from an orifice k, in the time t, is equal to twice the volume of a cylinder whose base is the area of the orifice, and whose height is a mean proportional between the depth of the orifice below the surface and the distance through which a body would fall in the time t.*

756. Let the vessel be now supposed to discharge itself, without receiving an additional supply of fluid: the quantity h in equation (433) must then be regarded as variable, and being replaced by $(a-z)$, that equation will become

$$u=\sqrt{[2g(a-z)]}.$$

This value of u substituted in (432) gives

$$dt=\frac{s'dz}{k\sqrt{[2g(a-z)]}};$$

or,

$$dt=\frac{s'dz}{k\sqrt{(2g)}\sqrt{(a-z)}}\;.\;.\;.\;.\;.\;(439).$$

The quantity s' represents the section of the vessel which corresponds to the upper surface of the fluid. This section will be a function of the variable z, and may be eliminated by means of the equation of the interior surface of the vessel. Thus, the value of s' in terms of z being introduced into equation (439) will render that equation susceptible of integration, and the relation between z and t will therefore become known. If we subtract the value of z thus obtained from the constant a, we shall obtain an expression for h in terms of t, which substituted in (435) will give, after integration, a relation between the time t and the quantity discharged Q.

757. Let us take, as an example, a vessel whose interior surface has the form of a paraboloid of revolution. This

surface being generated by the revolution of the parabolic arc AD (*Fig.* 253) about the vertical axis AB; if we denote by a the distance AB between the orifice and the surface of the fluid in its primitive position, by z the distance PB, and by y the ordinate PM, we shall have the relation,

$y^2=p(a-z)$ ..... *the equation of a parabola referred to its vertex* A.

Hence, if π represent the ratio of the circumference to the diameter, the area of the circle described with the radius PM will be expressed by $\pi y^2=\pi p(a-z)$; and consequently,

$$s'=\pi p(a-z) \; \ldots\ldots \; (440).$$

Let this value be substituted in (439), and we shall obtain

$$dt=\frac{\pi p}{k\sqrt{(2g)}}\times\frac{a-z}{\sqrt{(a-z)}}dz;$$

or, by reduction,

$$dt=\frac{\pi p}{k\sqrt{(2g)}}(a-z)^{\frac{1}{2}}dz.$$

758. For the purpose of integrating this equation, we make $a-z=x$; whence,

$$\int(a-z)^{\frac{1}{2}}dz=-\int x^{\frac{1}{2}}dx=-\tfrac{2}{3}x^{\frac{3}{2}}+\text{C}:$$

replacing x by its value, we have

$$\int(a-z)^{\frac{1}{2}}dz=-\tfrac{2}{3}(a-z)^{\frac{3}{2}}+\text{C};$$

and consequently,

$$t=-\tfrac{2}{3}\frac{\pi p}{k\sqrt{(2g)}}(a-z)^{\frac{3}{2}}+\text{C} \; \ldots\ldots \; (441).$$

The constant C is determined by making $z=0$ and $t=0$; this supposition gives

$$\text{C}=\tfrac{2}{3}\frac{\pi pa^{\frac{3}{2}}}{k\sqrt{(2g)}};$$

and the equation (441) can therefore be reduced to

$$t=\tfrac{2}{3}\frac{\pi p}{k\sqrt{(2g)}}[a^{\frac{3}{2}}-(a-z)^{\frac{3}{2}}].$$

To determine the quantity discharged in a given time, we find in this equation the value of

$$a-z=\left(a^{\frac{3}{2}}-\tfrac{3}{2}\frac{k\sqrt{(2g)}}{\pi p}t\right)^{\frac{2}{3}}$$

and substitute it for h in formula (435): we thus obtain the relation

$$Q=k\sqrt{(2g)}\int\left(a^{\frac{3}{2}}-\tfrac{3}{2}\frac{k\sqrt{(2g)}}{\pi p}t\right)^{\frac{1}{3}}dt.$$

This equation may be integrated by a process entirely similar to that adopted in finding the relation between z and t.

759. Let it be required to determine the time in which the water contained in a vessel having the form of a right cylinder will be discharged through an orifice in the bottom of the vessel. Let b represent the radius of a section of the cylinder by a plane perpendicular to its axis: then, $s'=\pi b^2$, and the equation (439), when applied to the present case, will give

$$t=\frac{\pi b^2}{k\sqrt{(2g)}}\times\int\frac{dz}{\sqrt{(a-z)}}.$$

Making $a-z=x$, then integrating the transformed equation, and replacing x by its value, we find

$$t=-\frac{2\pi b^2}{k\sqrt{(2g)}}\sqrt{(a-z)}+C.$$

The constant is determined as in the last example, by making $z=0$ and $t=0$: whence we deduce

$$t=\frac{2\pi b^2}{k\sqrt{(2g)}}[\sqrt{a}-\sqrt{(a-z)}]\;.\;.\;.\;.\;.\;(442).$$

The integral being taken between the limits $z=0$ and $z=a$, we find, for the time of emptying the vessel,

$$t=\frac{2\pi b^2}{k\sqrt{(2g)}}\sqrt{a}\;.\;.\;.\;.\;.\;(443).$$

If we suppose, as in Art. 749, that the orifice is a circle whose radius is equal to r, we shall have $k=\pi r^2$: this value reduces (443) to

$$t=\frac{2b^2}{\sqrt{(2g)}}\cdot\frac{\sqrt{a}}{r^2}.$$

By comparing this result with that obtained in Art. 753, it will appear that the time necessary for the entire discharge of the fluid when the vessel empties itself, is double that in which an equal quantity of fluid would flow through the same orifice if the vessel were kept constantly full.

760. The formulas (442) and (443) will serve as a guide in

the construction of a clepsydra, or water-clock. This instrument consists merely of a vessel from which the water is allowed to escape through an orifice in the bottom, and the intervals of time are measured by the depressions of the upper surface. Thus, if we wish the clock to run 12 hours, we reduce the time to seconds, which gives $12\times(60)^2$, or 12×3600; and by substituting this value of t in formula (443), we can then assume arbitrarily two of the three quantities k, b, and a. Let the values of k and b be assumed; that of a, the height of the clepsydra, will then result from formula (443).

To discover the manner in which this height should be divided in order that the superior surface of the fluid may be depressed through the several divisions of the scale in equal intervals of time, we deduce from equation (442) the value of $(a-z)$, which is

$$a-z=\left(\sqrt{a}-\frac{kt\sqrt{(2g)}}{2\pi b^2}\right)^2;$$

and by making t successively equal to 1 hour, 2 hours, 3 hours, &c., we can determine the corresponding values of $a-z$, which should be laid off from the bottom of the vessel. We can, however, readily discover the general law according to which the scale must be divided: for, since the vessel is supposed to discharge itself in 12 hours, if we make $t=12$ hrs., we shall have $a-z=0$; and consequently,

$$\sqrt{a}-\frac{k(12\text{ hrs.})\sqrt{(2g)}}{2\pi b^2}=0;$$

or,

$$\frac{(12\text{ hrs.})k\sqrt{(2g)}}{2\pi b^2}=\sqrt{a}.$$

When $t=11$ hrs., we have

$$\frac{t\,.\,k\sqrt{(2g)}}{2\pi b^2}=\frac{(11\text{ hrs.})k\sqrt{(2g)}}{2\pi b^2}=\tfrac{11}{12}\sqrt{a};$$

and therefore,

$$a-z=(\sqrt{a}-\tfrac{11}{12}\sqrt{a})^2=(\tfrac{1}{12})^2\times a.$$

In like manner, when $t=10$ hrs. we shall find

$$a-z=(\tfrac{2}{12})^2\times a.$$

Thus the successive values of $a-z$, which correspond to the several hours, will bear to each other the same relations as the terms in the series

$$(\tfrac{1}{12})^2,\ (\tfrac{2}{12})^2,\ (\tfrac{3}{12})^2,\ \&c.$$

These terms are to each other in the same ratio as the squares of the natural numbers 1, 2, 3, &c. Hence, if we divide the whole height a into 144 equal parts, and lay off from the bottom of the vessel distances which shall be equal respectively to 1, 4, 9, &c. of these parts, we shall obtain the points of division in the scale which will correspond to the upper surface of the fluid at the expiration of the several hours. The form of the vessel being prismatic, the figure of its base may be assumed arbitrarily.

761. When the surface of the fluid shall have arrived nearly at the bottom of the orifice, the quantity discharged will be influenced by the formation of a hollow tunnel, which is then found to be produced directly above the orifice: it is therefore advisable to employ only the first eleven divisions of the scale.

762. It usually occurs that the condition of the particles descending in vertical lines, and with velocities which are equal at every point of the same stratum, ceases to be fulfilled when the surface of the fluid has arrived within 4 or 5 inches of a horizontal orifice. The fluid particles then assume directions which are more or less inclined to the horizon, and the tunnel spoken of in the last article is then formed. When the orifice is found at a considerable depth, the upper surface of the fluid remains sensibly horizontal, and the tunnel above the orifice is no longer formed, in consequence of the greater velocity with which the fluid particles near the orifice are compelled to flow into the vacancy which has been left by those immediately preceding them.

763. This tunnel becomes much less perceptible when the orifice is formed in the side of a vessel. But when the upper surface of the fluid has nearly attained the level of the orifice, a slight depression on the side of the orifice begins to be observed.

764. This tendency of the fluid particles towards the orifice, occasioned by their sustaining less pressure in that direc-

tion, gives rise to a contraction in the jet of fluid, which, in issuing from the orifice, assumes the form of a truncated pyramid or cone, whose greater base corresponds to the orifice. This diminution in the size of the jet is called the *contraction of the vein.*

With a circular orifice, the smallest section of the fluid vein is found at a distance from the orifice equal to the radius of the orifice. Beyond this point the diameter of the section again increases, so that the entire jet has the form of two truncated cones which are united by their smaller bases.

765. The contraction of the vein likewise takes place when the orifice is pierced in the side of a vessel; but if the orifice be large, and be placed at a short distance below the surface of the fluid in the reservoir, the jet will be found to be more contracted in the vertical than in the horizontal direction.

766. When a conical tube whose interior surface corresponds to the form of the contracted vein is adapted to an orifice pierced in a thin plate, the quantity discharged is found to be very nearly the same as though the fluid issued directly through the orifice. Hence, we may regard the vessel as continued to the point at which the greatest contraction of the stream takes place, and consider the least section as forming the real orifice.

It is proved by experience, that the quantity actually discharged may be deduced from that calculated according to the theory, by simply changing the value of the constant k. Thus, if we represent by Mk the area of the orifice which has been calculated from a knowledge of the quantity actually discharged, the theoretic formula

$$Q = k\sqrt{(2g)} \cdot t\sqrt{a}$$

must be modified by substituting Mk for k: we shall thus obtain, for the actual discharge,

$$Q = Mk\sqrt{(2g)} \cdot t\sqrt{a}.$$

767. When the orifices are pierced in thin plates, the ratio M is found to be independent of the size of the orifice, and of its depth below the surface, provided that depth be not very small. Hence, if we represent by Q′ the quantity discharged

from an orifice k' at the depth a', we shall have the proportion

$$Q : Q' :: Mk\sqrt{(2g)}\,.\,t\sqrt{a} : Mk'\sqrt{(2g)}\,.\,t\sqrt{a'};$$

and we therefore conclude, that the quantities discharged from two such orifices are to each other as the products of the areas of those orifices, and the square roots of their depths.

768. The number M has been found by Bossut to be about 0.62, and the orifice k must therefore be multiplied by this fraction, in order that the quantity given by the formula may correspond with the results of experiment. Thus, the corrected expression for the quantity discharged will be

$$Q=(0.62)k\sqrt{(2g)}\,.\,t\sqrt{a}.$$

This formula is alike applicable, whether the orifice be pierced in the side or bottom of a vessel.

769. When the vessel is allowed to empty itself, the circumstances of the discharge become very complicated after the upper surface of the fluid has fallen to within a short distance of the orifice. If, however, we only consider the expenditure previous to the arrival of the upper surface within a few inches of the orifice, the same correction may be applied to formula (439), which will thus become

$$dt=\frac{s'dz}{(0.62)k\sqrt{(2g)}\sqrt{(a-z)}},$$

and will serve to determine the time necessary for a given quantity of fluid to be discharged.

770. In applying the preceding correction to the theoretical discharge, it has been supposed that the orifice was pierced in a thin plate: when a similar orifice is pierced in a thick plate, the quantity discharged is found to be considerably greater. Hence it occurs, that when the fluid is discharged through a thick plate, or through a cylindrical tube applied to the orifice, the coefficient 0.62, which has been employed in calculating the discharge through a thin plate, is no longer applicable. In this case the fluid adheres to the sides of the tube, and the contraction of the stream is in a great measure avoided. The lengths of such tubes, according to Bossut, should be at least twice the diameter of the orifice, in order that the contraction of the vein may be pre-

vented. There will however be a limit to the length, proper to be given to such tubes, since the friction of the fluid against the sides of the tube will necessarily increase with its length.

771. The quantities discharged by cylindrical tubes are proportional to the products of the orifices by the square roots of their depths, as in the case of apertures pierced in a thin plate; but the coefficient M, by which the area of the orifice must be multiplied for the purpose of reducing the theoretical discharge to that given by experiment, has been found by Bossut to be about $\frac{13}{16}$, or, more accurately, 0.81, when a short cylindrical tube is applied to the orifice. Thus, the formula (436), which serves to determine the quantity discharged from a reservoir in which the fluid is maintained at a constant height, will become, when corrected for the case of a cylindrical tube,

$$Q=(0.81)\sqrt{(2g)}\,.\,kt\sqrt{a}\,;$$

or, if we replace k by its value πr^2, r denoting the radius of its section, the formula may be written

$$Q=(0.81)\sqrt{(2g)}\,.\,\pi r^2 t\sqrt{a}.$$

772. When the vessel is supposed to empty itself by an orifice to which a cylindrical tube has been adapted, we can still employ the coefficient (0.81), provided we only consider the circumstances of discharge previous to the arrival of the upper surface of the fluid at such a level that the tunnel begins to be formed above the orifice.

773. By adapting tubes of different forms to an orifice pierced in the side or bottom of a vessel, the quantity of fluid discharged is generally found to be more or less increased.

The following table presents a view of the relative quantities discharged in some of the simplest cases.

1°. Theoretical discharge in a given time through an orifice pierced in a thin plate - - - -	1.00
2°. Actual discharge in the same time through the same orifice - - - - - - - - - - - - -	0.62
3°. Discharge through a cylindrical tube, whose length is equal to two diameters of the orifice - - - - - - - - - - - - - -	0.81

4°. Discharge through a conical tube having the form of the contracted vein, the larger base being regarded as the orifice - - - - -	0.62
5°. Discharge through the same tube, regarding the smaller base as the orifice - - - - - - -	1.00

Of the Motion of Water in Pipes.

774. Let AB (*Fig.* 254) represent a cylindrical pipe, by means of which the water contained in the reservoir R is transferred to the reservoir R′, and let it be supposed that the current has assumed a uniform motion: it is proposed to investigate a formula by means of which the quantity of water delivered at the point B, in a given time, may be estimated.

Let CC′D′D represent an elementary stratum of the fluid included between two consecutive transverse sections. Then, since the motion is supposed to have become uniform, the forces which tend to accelerate the motion of the element CD′ must be precisely equal to those which are exerted upon the element in a contrary direction. The force exerted upon CD′, urging it in a direction from A towards B, is the component of the weight of this element, in a direction parallel to the axis of the pipe: and the forces which urge it in an opposite direction are, 1°. the difference of the pressures exerted upon the faces CC′ and DD′; and, 2°. the resistance arising from the friction of the fluid against the sides of the pipe.

775. If p denote the mean pressure, referred to the unit of surface, in the section CC′, the corresponding pressure in the section DD′ will be expressed by $p+dp$, and if we denote by a the area of the transverse section of the pipe, the entire pressures upon the sections CC′ and DD′ will be respectively

$$adp, \quad a(p+dp).$$

These pressures being exerted in contrary directions, the elementary stratum CD′ will be acted upon by a force equal to their difference adp.

The resistance arising from the friction against the sides of the pipe will be directly proportional to the surface of the fluid in contact with the pipe, and will likewise be dependent upon the velocity of the current. Hence, if v denote the velocity, c the circumference of the section, and s the distance of the section CC′ from the extremity A, the distance CD will be expressed by ds, and the resistance experienced by the element CD′, in consequence of friction, will be

$$cds \, . \, \varphi(v);$$

in which $\varphi(v)$ represents a certain function of v, to be ascertained by experiment.

776. To obtain an expression for the force which acts in the direction from A towards B, we shall suppose the density of water to be equal to unity, and resolve the weight of the element which is expressed by $g \, . \, ads$ into two components, respectively parallel and perpendicular to the axis of the pipe. Then denoting by θ the angle included between the axis and the horizon, the component of the weight parallel to the axis will become $g \, . \sin \theta \, . \, ads$. But if z represent the vertical co-ordinate of the point C referred to A as an origin, dz will represent the difference of level of the points C and D; and we shall have

$$\frac{dz}{ds} = \sin \theta, \quad g \, . \sin \theta \, . \, ads = gadz$$

And since an equilibrium must subsist between this force and the forces exerted in an opposite direction, we have

$$gadz = adp + cds \, . \, \varphi(v);$$

and by integration,

$$gaz = ap + cs \, . \, \varphi(v) + \mathrm{C}.$$

To determine the value of the constant C, we suppose the pressure at the origin A to be equal to a known quantity P: we shall then have $p = \mathrm{P}$, $z = 0$, $s = 0$; and therefore

$$\mathrm{C} = -a\mathrm{P}.$$

Eliminating C between these two equations, we obtain

$$gaz = a(p - \mathrm{P}) + cs \, . \, \varphi(v).$$

And by taking the integral between the limits $s = 0$, and $s = \mathrm{AB} = l$, the entire length of the tube, denoting by P′ the pressure at the lower extremity, and by z' the co-ordinate of the point B, there results

$$g \, . \, az' = a(\mathrm{P}' - \mathrm{P}) + cl \, . \, \varphi(v) \, . \, . \, . \, . \, . \, (444).$$

777. It has been found by experiment, that the function $\varphi(v)$ may be expressed by two terms which are respectively proportional to the first and second powers of the velocity: thus, we shall have

$$\varphi(v)=bv+b'v^2\,;$$

b and b' representing constant quantities.

This value of $\varphi(v)$, being substituted in equation (444), gives

$$bv+b'v^2=\frac{agz'-a(\mathrm{P}'-\mathrm{P})}{cl}\,;$$

but if the diameter of the pipe be denoted by D, we shall have

$$\frac{a}{c}=\tfrac{1}{4}\mathrm{D}\,;$$

and therefore,

$$bv+b'v^2=\tfrac{1}{4}\mathrm{D}\frac{gz'-(\mathrm{P}'-\mathrm{P})}{l}.$$

778. The pressure P at the upper extremity of the pipe may be regarded without material error as that due to the depth EA of the point A below the surface of the fluid in the reservoir R. Strictly speaking, the pressure P is somewhat less than that due to the depth EA, since these pressures become equal only when the orifice is infinitely small (Art. 742); but the difference is inconsiderable when the velocity of the fluid is not great. In like manner, the pressure at the point B may be supposed due to the depth E′B of the point B below the surface of the fluid in the reservoir R′: hence, if h and h' represent the respective depths EA and E′B, we shall have (Art. 655) $\mathrm{P}=gh$, $\mathrm{P}'=gh'$; and by substitution we obtain

$$bv+b'v^2=\tfrac{1}{4}\mathrm{D}g\frac{z'-h'+h}{l}.$$

If we divide each member of this equation by g, and put, for brevity,

$$\frac{b}{g}=\alpha,\ \frac{b'}{g}=\beta,\ \frac{z'-h'+h}{l}=k,$$

we shall obtain

$$\alpha v+\beta v^2=\tfrac{1}{4}\mathrm{D}k\ \ldots\ldots\ (445).$$

The values of α and β may be regarded as known, since they result immediately from those of a and b, which are supposed to be determined by observation; and the value of k will likewise be given when the length of the pipe, the difference of level of its two extremities, and the difference of the pressures at those points are previously given. Hence, the velocity v in a pipe of a given diameter can be readily calculated.

779. The numerical values of α and β have been found by Prony to be

$$\alpha=0.00017, \quad \beta=0.000106;$$

and the preceding equation therefore becomes

$$0.00017v+0.000106v^2=\tfrac{1}{4}Dk.$$

If we neglect the first term, which is generally admissible when the velocity v is not extremely small, the formula will reduce to

$$v=48.56\sqrt{(Dk)}.$$

780. Let Q denote the quantity delivered at the point B in a second of time, and π the number 3.1416; we shall have

$$Q=\frac{\pi D^2}{4}v, \quad v=\frac{4Q}{\pi D^2};$$

and by substituting this value of v in equation (445), there results

$$\alpha\frac{4Q}{\pi D^2}+\beta\frac{16Q^2}{\pi^2 D^4}=\tfrac{1}{4}Dk;$$

or, if we neglect the term containing the first power of v, and make $\frac{64\beta}{\pi^2}=\beta'^2$, we shall obtain

$$Q=\frac{1}{\beta'}\sqrt{(D^5k)}.$$

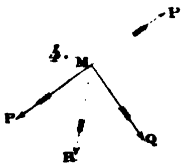
The numerical value of $\frac{1}{\beta'}$ is 38.12; and the formula therefore reduces to

$$Q=38.12\sqrt{(D^5k)}.$$

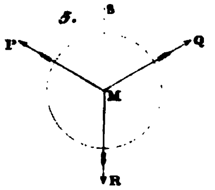
In this investigation the dimensions are supposed to be expressed in English feet.

THE END.

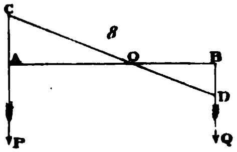
4.



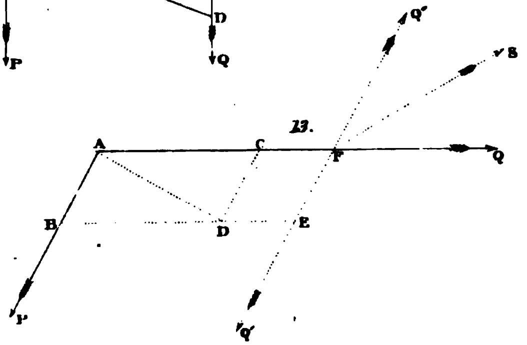
5.

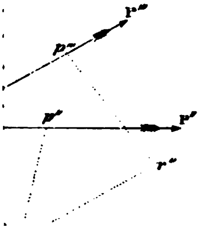


8

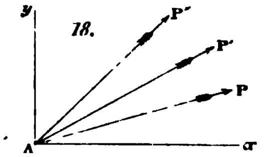


13.

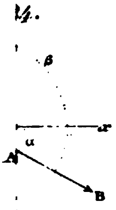




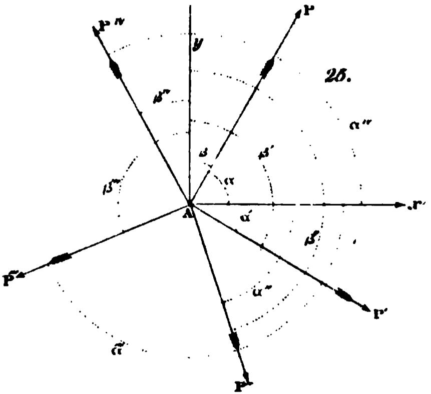
18.



4.

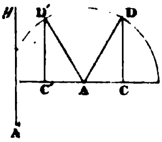


25.

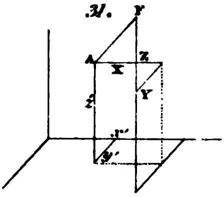


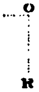
Prud'homme Sc. N.Y.

30.

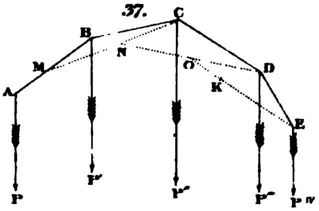


31.

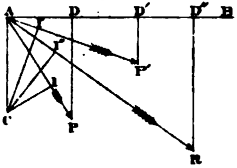


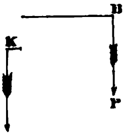


37.

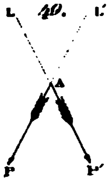


48.

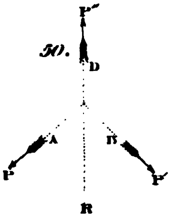




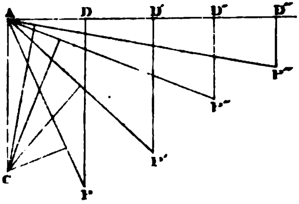
40.



50.



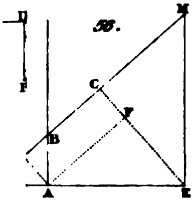
51.

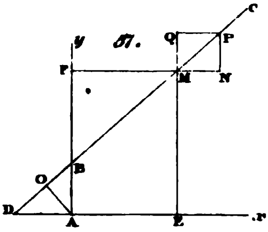


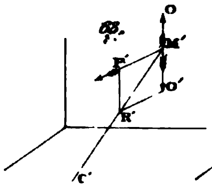
Prudhomme Sc. N.Y.

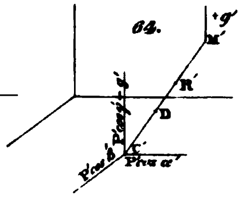
Pl. 3d

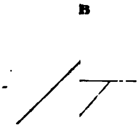
m N.Y.

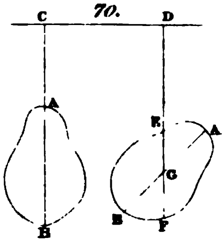

56.

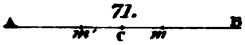

57.

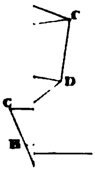

63.

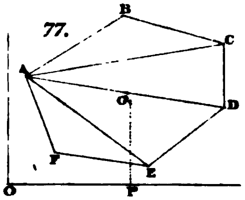

64.

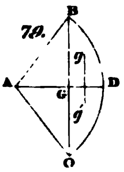


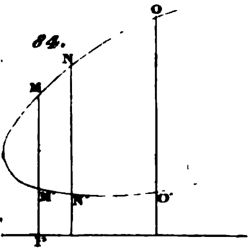

70.

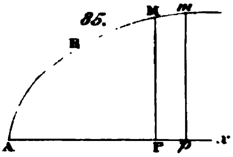

71.

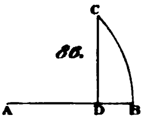



77.


79.

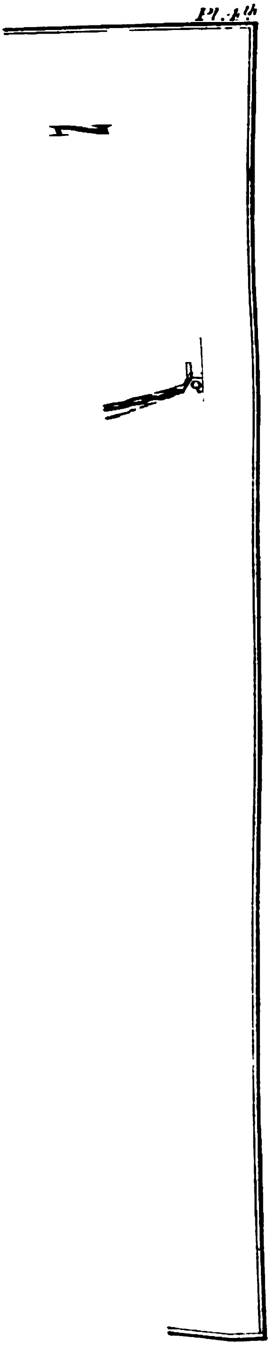

84.


85.

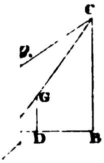

86.

Prud'homme sc. N.Y.

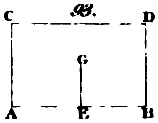
Pl. 4th



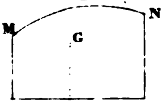
Pl. 4th

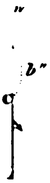


93.

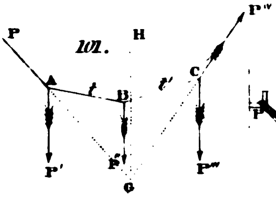


94.

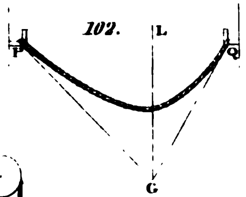




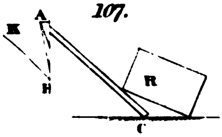
101.



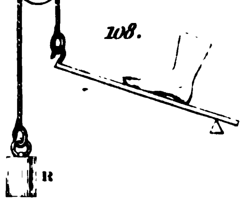
102.

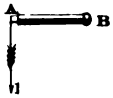


107.

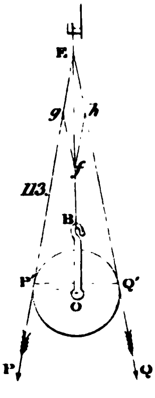


108.

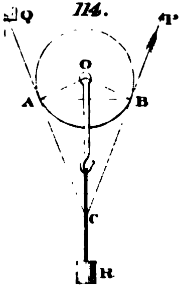


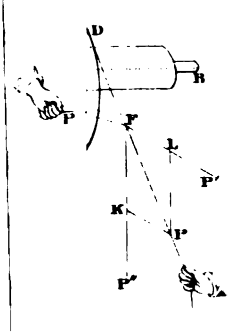


113.

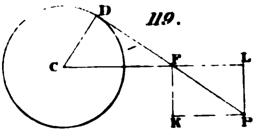


114.





119.

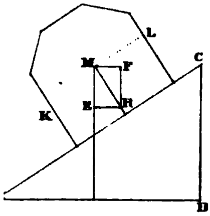


Pl. 5th

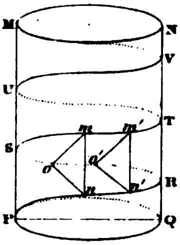
Sc N.Y.

Pl. 5th

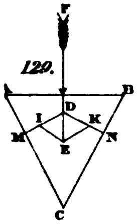
124.



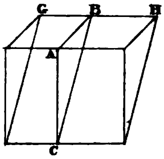
126.



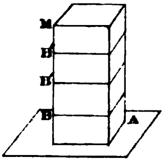
129.



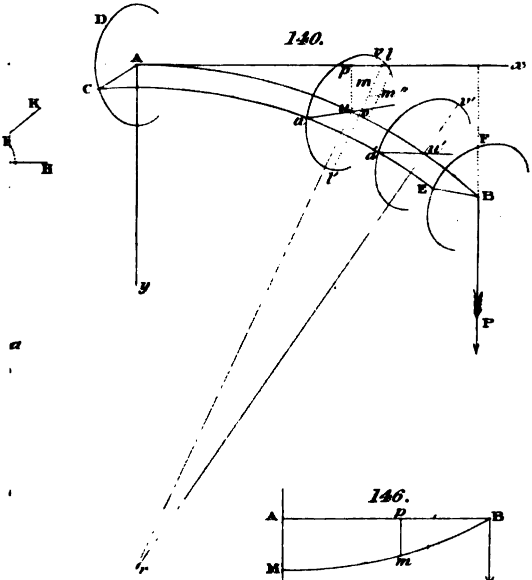
130.



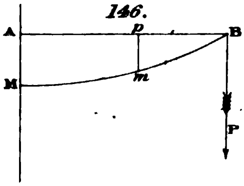
131.



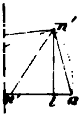
140.



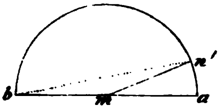
146.



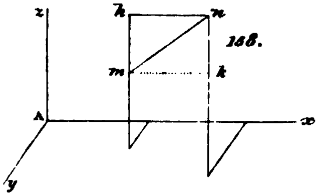
E. Prudhomme Sc. N.Y.



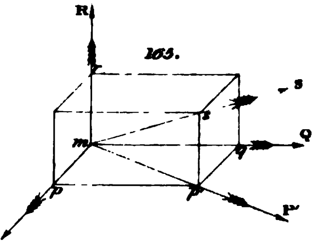
163.



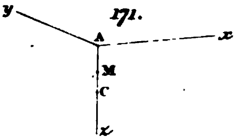
168.



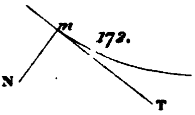
165.



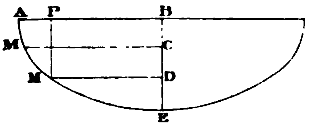
171.



172.

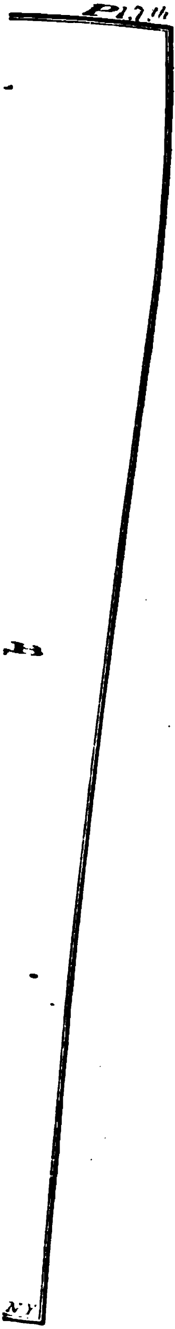


178.

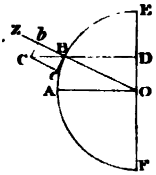


E. Prud'homme Sc.

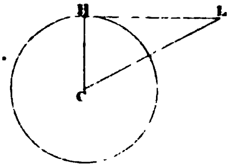
Pl. 7th



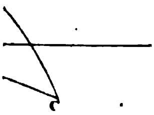
185.



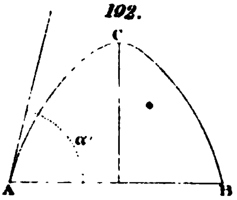
186.



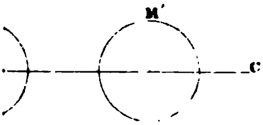
191.



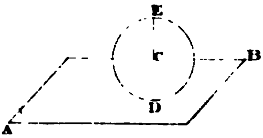
192.



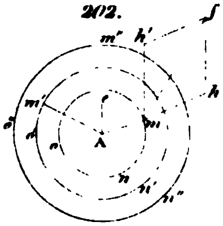
196



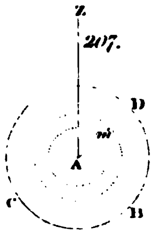
197.



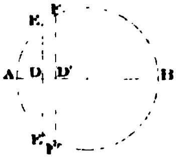
202.



207.



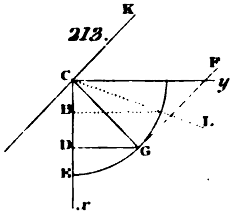
208.



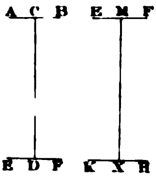
E. Prudhomme Sc. N.Y.

8th

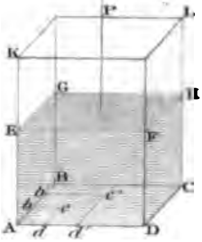
213.



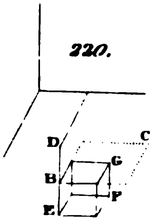
214.

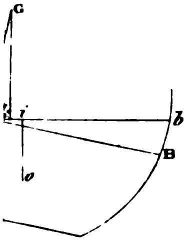


219.

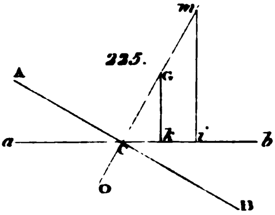


220.

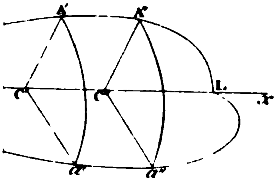




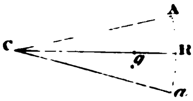
225.



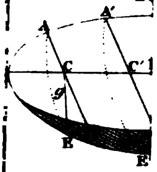
229.



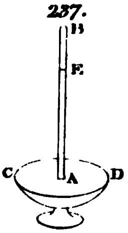
230.



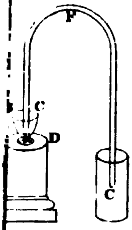
E. Prudhomme Sc. N.Y.



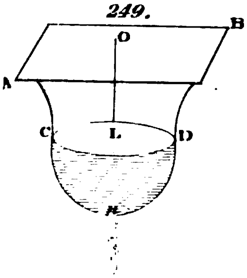
237.

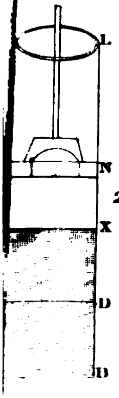


238.

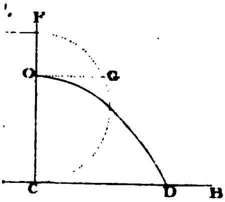


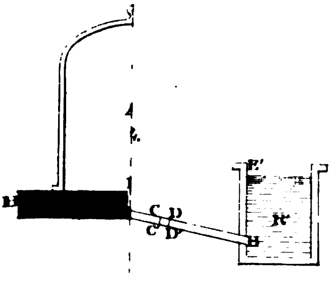
249.





243





E. Prudhomme Sc. N.Y.

JLF